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# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1344

ON THE THEORY OF THE TURBULENT BOUNDARY LAYER

By J. Rotta

Translation of "Über die Theorie der turbulenten Grenzschichten."  
Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung  
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By J. Rotta

## INTRODUCTION

(By A. Betz)

In view of the high specialization of scientific research many papers, basically important for further progress, are of interest only for a relatively small circle of close colleagues. In normal times, such reports nevertheless could be published in scientific periodicals without difficulty. The periodicals published papers from various fields and thus offered to a relatively large circle of readers sufficiently valuable material. Today, this procedure is faced with considerable difficulties which can be traced back to two main reasons: Scientific work has developed enormously so that periodicals had to be greatly increased in number and volume. Thus, on one hand, it takes the reader a great deal of time to follow the literature of his special field; on the other hand, subscription to periodicals represents a heavy financial burden. In addition, almost all scientists, especially in Germany, are greatly impoverished and can no longer carry the increasing financial load; the sale of periodicals is thereby reduced and the costs rise still further.

Looking for a way out of this difficulty I thought it desirable to relieve the periodicals, first of all, of reports which address only a relatively small circle of interested parties and yet, to be understandable, have to be somewhat extensive. For such reports the considerable cost expenditure required for issue of a good periodical does not pay; such expenditure is in order only in case of a correspondingly large circulation. In order to acquaint the few specialists with such reports and to make those reports accessible for later need, one can economically recommend only a reproduction method which is relatively cheap also in case of small circulation. On the basis of these considerations, I have decided to print such reports which originate in the Max-Planck-Institute for flow research and, also, a few older reports from this field, which are no longer available by Rota printing method, and to edit them in informal sequence as communications of the institute. Herewith, I give to our colleagues the first issue of these communications. May it fulfill the tasks described.

Göttingen, March 1950.  
Albert Betz

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\*"Über die Theorie der turbulenten Grenzschichten." Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung (Göttingen), No. 1, 1950.

## OUTLINE

## SURVEY

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## SURVEY

As a rule, a division of the turbulent boundary layer is admissible: a division into a part near the wall, where the flow is governed only by the wall effects, and into an outer part, where the wall roughness and the viscosity of the flow medium affects only the wall shearing stress occurring as boundary condition but does not exert any other influence on the flow. Both parts may be investigated to a large extent independently. Under certain presuppositions there result for the outer part "similar" solutions. The theoretical considerations give a cue how to set up, by appropriate experiments and their evaluation, generally valid connections which are required for the approximate calculation of the turbulent boundary layer according to the momentum and energy theorem.

## SYMBOLS

$x, y, z$  coordinates ( $x, z$  parallel to the wall;  $y$  perpendicular distance from the wall)

$\rho$  density  
 $\nu$  kinematic viscosity

} of the flow medium

Velocities and pressures, stresses:

$U, V, W$  velocity components of basic flow (average values in time) ( $U$  in  $x$ -direction,  $V$  in  $y$ -direction,  $W$  in  $z$ -direction)

$U_1$  flow velocity in  $x$ -direction, outside of the boundary layer

$u, v, w$  components of the turbulent fluctuation velocities

$p$  mean value in time of the static pressure

$\tilde{p}$  fluctuations of the static pressure

$\sigma_x, \sigma_y, \sigma_z$  normal stresses

} (mean values in time)

$\tau_{xy}, \tau_{xz}, \tau_{yz}$  shearing stresses

(in section 3 following  $\tau_{xy}$  is written  $\tau_{xy} = \tau$ )

$\tau_0$  wall shearing stress

$v^* = \sqrt{\tau_0 / \rho}$  shearing-stress velocity

$c_f' = 2(v^* / U_1)^2$  local friction coefficient

Turbulence quantities:

$E$  kinetic turbulence energy (per unit mass)

$S$  energy dissipation (per unit mass)

$\rho Q_x, \rho Q_y, \rho Q_z$  components of the energy diffusion (energy flow per unit time and unit area) (in section 3 following  $Q_y$  is written  $Q_y = Q$ )

} mean values in time

$D = \int_0^\infty S \, dy$  dissipation function

- $l$  characteristic length of the large turbulence elements  
(statement  $l = \kappa y$ ;  $\kappa \sim 0.4$ )
- $k, k_q, c$  dimensionless factors according to equations (3.16),  
(3.17), (3.18). (Characterized for the universal  
boundary-layer flow in the range  $\delta_w \leq y \ll \delta$  by the  
index "0")

Thicknesses of the boundary layer:

- $\delta$  total thickness
- $\delta_w$  thickness of the sublayer directly affected by the  
viscosity and the wall roughness

$$\delta_1 = \int_0^\infty (1 - U/U_1) dy \quad \text{displacement thickness}$$

$$\delta_2 = \int_0^\infty (U/U_1)(1 - U/U_1) dy \quad \text{momentum-loss thickness}$$

$$\delta_3 = \int_0^\infty (U/U_1) \left[ 1 - (U/U_1)^2 \right] dy \quad \text{energy-loss thickness}$$

$$\left. \begin{aligned} H_{12} &= \delta_1/\delta_2 \\ H_{32} &= \delta_3/\delta_2 \end{aligned} \right\} \text{form parameters}$$

Similar solutions:

- $\psi$  stream function
- $\eta = y/x$  dimensionless coordinate
- $m$  exponent of the law prescribed at the outer edge (eq. (5.4))
- $Re_x = \frac{U_1 x}{\nu}$  Reynolds number formed with the coordinate  $x$

Empirical boundary-layer profiles:

$$\left. \begin{aligned} I_1 &= \int_0^\infty \left[ \frac{U_1 - U}{v^*} \right]^2 d\left(\frac{yv^*}{\delta_1 U_1}\right) \\ I_2 &= \int_0^\infty \left[ \frac{U_1 - U}{v^*} \right]^3 d\left(\frac{yv^*}{\delta_1 U_1}\right) \end{aligned} \right\} \text{form parameters}$$

C constant of the velocity profile near the wall (eq. (4.8))

K constant of the outer velocity profile (eq. (6.8))

$Re_1 = \frac{U_1 \delta_1}{v}$  Reynolds number formed with the displacement thickness  $\delta_1$

$$B = \frac{U_1}{v^*} - \frac{1}{\kappa} \ln Re_1 = C + K$$

A form parameter in equation (6.12)

$$G = D/v^{*3} - \frac{1}{\kappa} \ln Re_1$$

## 1. INTRODUCTION

For evaluation of the flow conditions about a body and in particular for estimation of its flow drag, the behavior of the flow layer bordering on the body, which may be either laminar or turbulent, is of very high importance. Whereas, for the laminar boundary layers, the physical relations have been clarified and the mathematical problems, too, have been worked out sufficiently to have calculation methods at disposal which are serviceable in practice, there are, for the turbulent boundary layers, above all still problems of a physical kind to be solved.

From the basic hydrodynamic equations, one may derive relations for the time averages of the flow quantities in turbulent boundary layers which are similar to those valid for laminar boundary layers. Furthermore, a relation for the energy balance of the turbulent movement is at disposal which was given first by L. Prandtl (ref. 1). In spite of these equations, an exact calculation of the turbulent boundary layers

is not possible since one has not yet succeeded in setting up formulas for essential processes in the mechanism of the turbulent movement. Thus the question arises whether there is, under such circumstances, any sense in a discussion of the boundary-layer equations. Actually, however, a few statements are possible on the basis of the means at disposal, if one considers the following two empirical facts which may be regarded as absolutely certain today:

(1) The total processes are affected by the kinematic viscosity  $\nu$  and the geometrical properties of the wall (wall roughness) only in a very thin layer in the neighborhood of the wall; in the remaining domain of the boundary layer, the flow appears to be practically independent of the viscosity and the wall roughness. If the thickness of the layer in which these influences are effective is called  $\delta_w$  and the thickness of the boundary layer  $\delta$ , one has therefore, as a rule,  $\delta_w \ll \delta$ .

(2) Because of  $\delta_w \ll \delta$  one may expect the conditions for wall distances  $y \leq \delta_w$  to be independent of the flow conditions at the outer edge of the boundary layer ( $y \rightarrow \delta$ ). Inside of the layer  $\delta_w$  there exists, therefore, a velocity law affected solely by the geometrical properties of the wall; the wall shearing stress  $\tau_0$  represents the essential parameter.

With these assumptions, one may separate the influence of the kinematic viscosity and that of the wall properties from the other influences. Thereby, it becomes not only possible to discuss various properties of the turbulent boundary layers but also to determine empirically, with the aid of similarity relations, from suitable measurements, the quantities needed for the development of approximation methods for calculation of turbulent boundary layers. Such series of measurements with all required quantities are not available in a desirable form at present; however, it is possible to perform them with today's test techniques.

In the following sections only flows of an incompressible fluid are considered which are steady on the average.

## 2. ENERGY BALANCE OF TURBULENT FLOWS

Since following the motion to the last detail is not possible in turbulent flows, a statistical treatment must be applied. The flow quantities will be expressed by arithmetical mean values, and the averaging in time will be simplest where one deals on the average with stationary flows. The time-averaged velocity with the components  $U$ ,  $V$ , and  $W$  in  $x$ -,  $y$ -, and  $z$ -direction is called basic flow. Superimposed

on it is the disordered turbulence motion fluctuating with time with the components  $u$ ,  $v$ , and  $w$ , which is always three-dimensional even when the basic flow may be regarded as two-dimensional. The velocity fluctuations cause fluctuations of the static pressure which will be denoted by  $\tilde{p}$ , whereas the time average of the static pressure is expressed by  $\bar{p}$ .

If one sets up, for the purpose of theoretical treatment, the Navier-Stokes equations of motion for turbulent flows and performs the time averaging, a few mean-value expressions formed from the fluctuation velocities, which are to be regarded as new unknowns, remain in the equations. In the search for further relations, in order to establish a mutual connection between these unknowns as well as a connection with the other flow quantities, one can derive from the Navier-Stokes differential equations further equations which partly convey very interesting insight into the turbulent flow phenomena. It is not the purpose of this report to discuss this more closely; but an important equation among those mentioned above, which describes the balance of the kinetic energy contained in the velocity fluctuations and for that reason is physically the most graphic one, is utilized subsequently to a very great extent. Since it is not yet to be found in exact form in literature, it will be given herein for general three-dimensional flows.

With the occurrence of shearing and tensile or compressive stresses, kinetic energy is withdrawn from a basic flow which partly reappears as kinetic energy of the disordered turbulence motion. Let the time average of this turbulence energy, referred to the unit mass, be

$$E = \frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{2} \quad (2.1)$$

with the bars signifying the time averaging. Owing to the viscosity of the fluid, kinetic energy is withdrawn continually from the basic flow as well as from the turbulence motion and is converted into heat (dissipation  $S$ ); moreover, because of the turbulence motion in general, an exchange of turbulence energy takes place between various points of the flow space. If one deals with nonhomogeneous turbulence, which is mostly the case, these exchange processes do not balance and an energy transport  $\rho Q$  occurs which one could compare to a diffusion process. An energy balance for the coordination of these single effects (as first formulated in this manner by L. Prandtl (ref. 1)) expresses that the sum of these contributions equals the total (substantial) variation of the turbulence energy. For a three-dimensional basic flow with the



components  $U$ ,  $V$ , and  $W$  this energy balance reads in the stationary case quite generally:

$$\rho \left[ U \frac{\partial E}{\partial x} + V \frac{\partial E}{\partial y} + W \frac{\partial E}{\partial z} \right] =$$

Total variation of the turbulence energy

$$= \sigma_x \frac{\partial U}{\partial x} + \sigma_y \frac{\partial V}{\partial y} + \sigma_z \frac{\partial W}{\partial z} + \tau_{xy} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) + \tau_{xz} \left( \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right) + \tau_{yz} \left( \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right)$$

Energy withdrawn from the basic flow

$$-\rho S \quad - \rho \left[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] \quad (2.2)$$

Dissipation      Energy diffusion

Therein

$$\left. \begin{aligned} \sigma_x &= \rho \left( 2\nu \frac{\partial U}{\partial x} - \overline{u^2} \right) \\ \sigma_y &= \rho \left( 2\nu \frac{\partial V}{\partial y} - \overline{v^2} \right) \\ \sigma_z &= \rho \left( 2\nu \frac{\partial W}{\partial z} - \overline{w^2} \right) \end{aligned} \right\} \quad (2.3)$$

are the time-averaged normal stresses and

$$\left. \begin{aligned} \tau_{xy} &= \rho \left[ v \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - \overline{uv} \right] \\ \tau_{xz} &= \rho \left[ v \left( \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right) - \overline{uw} \right] \\ \tau_{yz} &= \rho \left[ v \left( \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right) - \overline{vw} \right] \end{aligned} \right\} \quad (2.4)$$

are the time-averaged shearing stresses. For the dissipation the expression

$$\begin{aligned} S = v & \left[ 2 \left( \frac{\partial U}{\partial x} \right)^2 + 2 \left( \frac{\partial V}{\partial y} \right)^2 + 2 \left( \frac{\partial W}{\partial z} \right)^2 + \left( \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} \right)^2 + \right. \\ & \left( \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right)^2 + 2 \overline{\left( \frac{\partial u}{\partial z} \right)^2} + 2 \overline{\left( \frac{\partial v}{\partial y} \right)^2} + 2 \overline{\left( \frac{\partial w}{\partial z} \right)^2} + \\ & \left. \overline{\left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2} + \overline{\left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2} + \overline{\left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2} \right] \end{aligned} \quad (2.5)$$

is valid, and the components  $Q_x$ ,  $Q_y$ , and  $Q_z$  of the energy diffusion have the following form

$$\left. \begin{aligned} Q_x &= -v \left( \frac{\partial E}{\partial x} + \frac{\partial \overline{u^2}}{\partial x} + \frac{\partial \overline{uv}}{\partial y} + \frac{\partial \overline{uw}}{\partial z} \right) + u \left( \frac{u^2 + v^2 + w^2}{2} + \frac{\tilde{p}}{\rho} \right) \\ Q_y &= -v \left( \frac{\partial E}{\partial y} + \frac{\partial \overline{vu}}{\partial x} + \frac{\partial \overline{v^2}}{\partial y} + \frac{\partial \overline{vw}}{\partial z} \right) + v \left( \frac{u^2 + v^2 + w^2}{2} + \frac{\tilde{p}}{\rho} \right) \\ Q_z &= -v \left( \frac{\partial E}{\partial z} + \frac{\partial \overline{wu}}{\partial x} + \frac{\partial \overline{wv}}{\partial y} + \frac{\partial \overline{w^2}}{\partial z} \right) + w \left( \frac{u^2 + v^2 + w^2}{2} + \frac{\tilde{p}}{\rho} \right) \end{aligned} \right\} \quad (2.6)$$

Equation (2.2), the derivation of which would be too lengthy here, is attained by addition of the three Navier-Stokes equations of motion after they have been multiplied by  $u$ ,  $v$ , and  $w$ , respectively. By application of the continuity equation and several transformations one finally obtains, after having formed the mean values, the form (2.2) with the expressions (2.5) and (2.6).

After this explanation which applies quite generally for turbulent flows, we shall deal with the special problem of turbulent boundary-layer flow.

### 3. EQUATIONS OF THE TURBULENT BOUNDARY LAYERS

#### 3.1. Basic equations

In the following, the  $x$ -axis is assumed to lie parallel to the wall and  $y$  to be the vertical distance from the wall. The three-dimensional turbulence motion with the components  $u$ ,  $v$ ,  $w$  is assumed to be superimposed on the components  $U$  and  $V$  of the two-dimensional basic flow in  $x$ - and  $y$ -direction. The boundary-layer equation for a two-dimensional flow along a plane wall resulting from the Navier-Stokes equations of motion then reads with the simplifications introduced by L. Prandtl<sup>1</sup>

$$\rho \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \quad (3.1)$$

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<sup>1</sup>The theory developed by L. Prandtl in 1904 at first for laminar boundary layers starts from the fact that the processes producing the friction drag take place in a very thin layer on the body. Accordingly, one may assume, for simplification of the problem, that the velocity component  $V$  normal to the wall is small compared with the component  $U$  acting parallel to the wall; furthermore, the static pressure  $p$  may be assumed to be independent of the wall distance. An estimation of the order of magnitude then indicates which terms in the equations may be neglected. In case of turbulent boundary layers the mean value  $p$  is influenced by the velocity fluctuations ( $p = p_0 - \overline{\rho v^2}$ ;  $p_0 = p$  for  $y = 0$ ); in the derivation with respect to  $x$  this slight influence is partly compensated by the term  $-\overline{\rho u^2}/\partial x$  neglected in equation (3.1) so that  $\partial p/\partial x$  may be regarded as independent of  $y$ .

Furthermore, one uses the continuity equation for the basic flow

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (3.2)$$

The continuity condition must of course be satisfied also by the fluctuating motion  $u$ ,  $v$ ,  $w$  which is already taken into consideration in the following formulas. In case of turbulent boundary-layer flows, there applies for the shearing stress  $\tau$  in the  $xy$ -plane the expression

$$\tau = \rho \left( \nu \frac{\partial U}{\partial y} - \overline{uv} \right) \quad (3.3)$$

Herein  $\overline{uv}$  is the time mean value of the product of the fluctuation components  $u$  and  $v$  acting vertically to one another. The expression  $-\rho \overline{uv}$  is also denoted as Reynolds stress.

We now include into our considerations as a further equation the energy balance of the turbulent flows given in section 2. For steady two-dimensional boundary layers expressions (2.2), (2.5), and (2.6) are simplified, under the customary assumptions<sup>2</sup> to

$$\rho \left( U \frac{\partial E}{\partial x} + V \frac{\partial E}{\partial y} \right) = \tau \frac{\partial U}{\partial y} - \rho S - \rho \frac{\partial Q}{\partial y} \quad (3.4)$$

$$S = \nu \left[ \left( \frac{\partial U}{\partial y} \right)^2 + 2 \overline{\left( \frac{\partial u}{\partial x} \right)^2} + 2 \overline{\left( \frac{\partial v}{\partial y} \right)^2} + 2 \overline{\left( \frac{\partial w}{\partial z} \right)^2} \right. \\ \left. + \overline{\left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2} + \overline{\left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2} + \overline{\left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2} \right] \quad (3.5)$$

$$Q = -\nu \left( \frac{\partial E}{\partial y} + \frac{\partial v^2}{\partial y} \right) + \nu \left( \frac{u^2 + v^2 + w^2}{2} + \frac{\tilde{p}}{\rho} \right) \quad (3.6)$$

---

<sup>2</sup>See also footnote 1.  $\tau$  is put equal to  $\tau_{xy}$  and  $Q$  is put equal to  $Q_y$ . The indices may be omitted here as before in equation (3.1) and later on, since a confusion is quite impossible.

The boundary-layer equations (3.1), (3.2), and (3.4) have to satisfy the following boundary conditions:

$$\begin{aligned} y = 0: \quad U &= 0; \quad V = 0; \quad \overline{u^2} = \overline{v^2} = \overline{w^2} = 0 \\ y \rightarrow \delta: \quad U &\rightarrow U_1; \quad \overline{u^2} \rightarrow 0; \quad \overline{v^2} \rightarrow 0; \quad \overline{w^2} \rightarrow 0 \end{aligned} \quad (3.7)$$

$U_1$  is the velocity outside of the boundary layer which is assumed to be prescribed as a function of  $x$ .

These relations are valid under the assumption of a sufficiently smooth wall. For uneven walls the formulation is considerably more complicated. In section 4 we shall revert to the treatment of rough walls where unevennesses of a certain mean magnitude are statistically distributed over the surface.

### 3.2. Momentum and Energy Theorem

If one introduces the quantities

$$\text{Displacement thickness } \delta_1 = \int_0^\infty \left(1 - \frac{U}{U_1}\right) dy \quad (3.8)$$

$$\text{Momentum-loss thickness } \delta_2 = \int_0^\infty \frac{U}{U_1} \left(1 - \frac{U}{U_1}\right) dy \quad (3.9)$$

and wall shearing stress  $\tau_0 = \tau$  for  $y = 0$ , one may derive from equations (3.1) and (3.2) the momentum equation

$$\frac{d}{dx} (U_1^2 \delta_2) + U_1 \delta_1 \frac{dU_1}{dx} = \frac{\tau_0}{\rho} \quad (3.10)$$

which has proved to be advantageous for the approximated integration of the boundary-layer equation, and which has the same form for turbulent as for laminar boundary layers.

By integration over  $y$  one may develop from equation (3.4) a corresponding energy equation. The partial integration of the left side of equation (3.4) yields, with use of equation (3.2)

$$\int_0^{\infty} \left( U \frac{\partial E}{\partial x} + v \frac{\partial E}{\partial y} \right) dy = \frac{d}{dx} \int_0^{\infty} UE dy \quad (3.11)$$

Furthermore there applies for the basic flow the relation to be derived from equation (3.1) (cf. the reports by K. Wieghardt, ref. 2)

$$\int_0^{\infty} \tau \frac{\partial U}{\partial y} dy = \rho \frac{d}{dx} \int_0^{\infty} \frac{U}{2} (U_1^2 - U^2) dy \quad (3.12)$$

The diffusion term  $\frac{\partial Q}{\partial y}$  in equation (3.4) disappears in the integration since the components  $u$ ,  $v$ ,  $w$  tend toward zero for  $y = 0$  as well as for  $y \rightarrow \delta$ . After introduction of the energy-loss thickness

$$\delta_3 = \int_0^{\infty} \frac{U}{U_1} \left[ 1 - \left( \frac{U}{U_1} \right)^2 \right] dy \quad (3.13)$$

and of the dissipation function

$$D = \int_0^{\infty} S dy \quad (3.14)$$

one then obtains as the energy theorem

$$\underbrace{\frac{1}{2} \frac{d}{dx} (U_1^3 \delta_3)}_{\text{Energy flow loss of the basic flow per unit length}} = \underbrace{D}_{\text{Dissipation}} + \underbrace{\frac{d}{dx} \int_0^{\infty} UE dy}_{\text{Increase of the turbulent energy flow per unit length}} \quad (3.15)$$

This equation is significant for the behavior of turbulent boundary layers. The energy losses of the basic flow essentially are first converted into kinetic turbulent energy which in turn is transformed into

heat by friction; however, the conversion of the basic flow energy into turbulent energy, and the transformation of the turbulent energy into heat need not take place at the same location and at the same time. This state of affairs is expressed in equation (3.15). Very frequently the increase of the turbulence energy flow contributes only slightly to equation (3.15) — thus, for instance, in case of ordinary plate flow without pressure gradient — however, there are also cases where this term gains more significant influence besides the dissipation function.

### 3.3. Supplementary Considerations Regarding Boundary-Layer Turbulence

By qualitative considerations one obtains a survey of the connections still lacking between the quantities  $E$ ,  $\overline{uv}$ ,  $S$ , and  $Q$  occurring in the energy balance (eq. (3.4)). We shall limit ourselves especially to the region  $y \geq \delta_w$  where the viscosity of the flow medium may be regarded as arbitrarily small. First of all, the terms with  $\nu$  in equations (3.3) and (3.6) are hereby eliminated; whereas, in equation (3.5) the contribution  $\nu(\partial U/\partial y)^2$  of the basic flow to the dissipation becomes negligibly small.

The amount  $E$  of the kinetic energy of the turbulence, which is a quantity suitable for dimensional considerations, is composed of the contributions of a very large number of turbulence elements of many different orders of magnitude; however, there exist for turbulence two characteristic lengths. One is the characteristic length  $l$  for the dimensions of the large turbulence elements, which for boundary-layer flows is given approximately by the pertinent distance from the wall. The second is the diameter  $l_R$  of the smallest turbulence elements, which is determined by the quotient of the kinematic viscosity and the mean fluctuation velocity — thus, approximately by  $\nu/\sqrt{E}$ . For the following considerations, the fact is important that the kinetic energy  $E$  is contained chiefly in the large elements and that, therefore, the momentum and energy exchange phenomena are essentially caused by the large elements. If the viscosity is sufficiently small or, more accurately, if the Reynolds number  $\frac{\sqrt{E}l}{\nu}$  of the turbulence is sufficiently large, which is the case in the region  $y \geq \delta_w$ , we need in our considerations only to refer to the one characteristic length  $l$  which corresponds to the dimensions of the large elements to make the total effects of turbulence independent of the viscosity.

The apparent shearing stress  $\overline{p_{uv}}$  caused by the turbulent fluctuation velocities may be traced back to a momentum transport which can be expressed by the form used by L. Prandtl (ref. 1)

$$\frac{\tau}{\rho} = -\overline{uv} = k\sqrt{E}l \frac{\partial U}{\partial y} \quad (3.16)$$

wherein  $k$  is a dimensionless factor. The product  $k\sqrt{E}l$  represents an apparent kinetic viscosity ( $\text{cm}^2/\text{sec}$ ) or, respectively, an exchange quantity - concepts first introduced by J. Boussinesq and W. Schmidt.

By considerations similar to those on which equation (3.16) is based, one arrives for the energy transport  $Q$  caused by exchange phenomena at the expression introduced by L. Prandtl (ref. 1)

$$Q = -k_1\sqrt{E}l \frac{\partial E}{\partial y}$$

Since, however, the turbulence elements of different order of magnitude have different energy density, an energy diffusion takes place, not only when an energy gradient is present but is obviously possible also when the turbulence at adjacent locations differs only by the linear dimensions or by the structure (for instance, the energy spectrum). With this interpretation the expression

$$Q = -\frac{\partial(k_q l E^{3/2})}{\partial y} \quad (3.17)$$

is justified,<sup>3</sup> which originates by taking the exchange quantity  $k\sqrt{E}l$  for the energy transport under the differential sign. Prandtl's form is contained in expression (3.17) as a special case. Here again  $k_q$  is a dimensionless factor which like  $k$  in equation (3.16) is chiefly determined by the structure of the large turbulence elements.

In contrast, the energy dissipation is caused mainly by the small elements. If one combines again the influences dependent on the structure (that is, the spectrum) in a dimensionless factor  $c_1$ , one obtains according to equation (3.5), with omission of the contribution  $v(\partial U/\partial y)^2$ , the relation

$$S = \nu c_1 \frac{E}{l^2}$$

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<sup>3</sup>The expressions (3.16) and (3.17) cannot yet lay claim to full general validity. A further discussion of these questions is not possible within the scope of this report and will, therefore, be the subject of another publication. For the present problem, the expressions (3.16) and (3.17) are, at any rate, sufficient.



Herein  $c_1$  is determined chiefly by the small turbulence elements and is independent of the Reynolds number only for very small  $\frac{\sqrt{E}l}{\nu}$  (cf. ref. 3).

For sufficiently large Reynolds numbers, we may express the dissipation process as a wandering of the kinetic energy (taking its course independently of the viscosity) from larger to smaller elements whereby, however, the energy content of the turbulence motion does not change. The extent of transformation into heat, occurring almost exclusively in the smallest turbulence elements, is guided by the amount of energy supplied to them by the larger elements (cf. the reports by C. F. v. Weizsäcker (ref. 4) and W. Heisenberg (ref. 5)). Thus for large Reynolds numbers, one may replace in the given expression for  $S$  the kinematic viscosity by an apparent kinematic viscosity of the dimension  $\sqrt{E}l$  one then obtains the relation

$$S = c \frac{E^{3/2}}{l} \quad (3.18)$$

which is valid for  $y \geq \delta_w$ . The magnitude of the factor  $c$  appearing therein is determined mainly by the large elements.

The dimensionless quantities  $k$ ,  $k_q$ , and  $c$ , (appearing in equations (3.16), (3.17), and (3.18)) which depend on the structure of the turbulence, will generally assume amounts differing from point to point; however, for sufficiently large Reynolds numbers, they are independent of the kinematic viscosity. Their calculation presupposes complete theoretical mastery of the statistic behavior of the turbulence motion. This goal, for turbulence research, however, is still far remote. For the following investigations,  $k$ ,  $k_q$ , and  $c$  are therefore introduced formally as functions of the location although without selection of special statements. However, it will be possible to assume offhand that they are continuous functions and do not become infinite at any point.

For the characteristic length  $l$  in equations (3.16), (3.17), and (3.18) for the large turbulence elements, there is, with consideration of the regions near the wall, the expression

$$l = \kappa y \quad (3.19)$$

of advantage where  $\kappa$  is a general constant. In this form, the length  $l$  is in the region  $\delta_w \leq y \ll \delta$  identical with the mixing length introduced by L. Prandtl (ref. 6). For the sake of simplicity, the expression (3.19) is used for the entire boundary layer, although the dimensions of the large turbulence elements for larger distances from the wall no longer increase in proportion to  $y$ . The deviations between the actual dimensions of the large elements and the expression (3.19) one may assume as taken into consideration in the factors  $k$ ,  $k_q$ , and  $c$ .

By introduction of the relations (3.16), (3.17), (3.18), and (3.19) into the boundary-layer equations, one may obtain a few statements regarding the behavior of the solutions; no limitation of the general validity seems to be connected with it.

#### 4. THE UNIVERSAL TURBULENT BOUNDARY-LAYER FLOW

For wall distances which are small compared with the boundary-layer thickness  $\delta$ , the shearing stress does not noticeably deviate from the value  $\tau_0$  of the wall shearing stress and the flow conditions are here practically independent of the pressure gradient  $\partial p / \partial x$ . It has already been mentioned in the introduction that the viscosity and the wall roughness exert an immediate influence on the flow phenomena only in a layer of the thickness  $\delta_w$  adjacent to the wall. If this thickness  $\delta_w$  is sufficiently small, there will certainly exist wall distances  $y$  larger than  $\delta_w$ , yet so small compared with the boundary-layer thickness  $\delta$  that in this region a universal boundary-layer flow results for which all flow quantities are determined by only two quantities which have physical dimensions, namely, the shearing stress velocity

$$v^* = \sqrt{\tau_0 / \rho} \quad (4.1)$$

and the absolute distance  $y$  from a plane of reference which practically coincides with the wall surface. This flow is influenced by the viscosity, the wall properties, and the pressure gradient  $\partial p / \partial x$  only insofar as they determine the magnitude of  $v^*$ . Aside from this influence, the flow in this region is not affected by either the outer boundary conditions or the wall properties and the viscosity. The velocity variation of the basic flow is prescribed for  $\delta_w \leq y \ll \delta$  by the known relation (ref. 6).

$$\kappa y \frac{\partial U}{\partial y} = v^* \quad (4.2)$$

Therein  $\kappa \sim 0.4$  is a universal constant which is identical with the value  $\kappa$  in equation (3.19).

Not only the velocity variation is known, however, but also important statements are possible concerning the turbulence energy and dissipation. Within the validity range of the universal boundary-layer flow no value of  $y$  is in any way distinguished. The structure of turbulence (energy spectrum, etc.) is therefore similar in all sections parallel to the wall<sup>4</sup>.

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<sup>4</sup>With the exception of the smallest turbulence elements.

Owing to this similarity and the equality of the shearing stress  $\tau_0 = -\rho \overline{uv}$ , the energy  $E$  has a value independent of the wall distance  $y$  so that the turbulence at different wall distances of this region differs only in its linear dimensions.

The factors  $k$ ,  $k_q$ ,  $c$  in expressions (3.16), (3.17), and (3.18) become for  $\delta_w \leq y \ll \delta$  generally valid constants which we shall emphasize by the subscript "0" ( $k_0$ ,  $k_{q0}$ ,  $c_0$ ). With  $-\overline{uv} = v^{*2}$  and  $l = \kappa y$  there follows from the relations (3.16) and (4.2)

$$E = \left( \frac{v^*}{k_0} \right)^2 \quad (4.3)$$

With this value, the expression (3.17) then yields for the energy transport  $Q$  caused by exchange a value also independent of  $y$

$$Q = -\kappa \frac{k_{q0}}{k_0^3} v^{*3} \quad (4.4)$$

which obviously corresponds to an energy flow in the direction toward the wall. Since, furthermore, under the presuppositions named, the terms on the left side of equation (3.4) become, in first approximation, small compared with the expressions on the right side and finally (because  $Q = \text{Const.}$ ) the last term on the right side disappears, the dissipation is, for  $y \ll \delta$ , equal to the energy withdrawn from the basic flow:

$$S = \frac{\tau_0}{\rho} \frac{\partial U}{\partial y} = v^{*2} \frac{\partial U}{\partial y} \quad (4.5)$$

Hence results with expressions (3.18), (3.19), (4.2), and (4.3) the relation

$$c_0 = k_0^3 \quad (4.6)$$

which like equation (4.3) was found by L. Prandtl (ref. 1); on the basis of measurements, the value of  $k_0$  was estimated to be  $k_0 = 0.56$ .

The existence of the universal boundary-layer flow in the region  $\delta_w \leq y \ll \delta$  suggests the division of the boundary-layer flow into a part near the wall ( $0 \leq y \ll \delta$ ) and an outer part ( $y > \delta_w$ ). The flows of both parts merge asymptotically into the universal boundary-layer flow: The flows of the first part with increasing  $y$ , those of the second with decreasing  $y$ . The advantage attained by this division is that one is now able to investigate the flow phenomena in each part separately with reduction of the influencing quantities (experimentally or theoretically) and to combine both parts, as occasion demands, since both have the same asymptotic variation at the point of junction.

For the part near the wall ( $0 \leq y \ll \delta$ ), there exists a velocity law of the general form (ref. 6)

$$U = v^* f\left(\frac{v^* y}{\nu}\right) \quad (4.7)$$

wherein the function  $f\left(\frac{v^* y}{\nu}\right)$  is dependent not only on  $\frac{v^* y}{\nu}$  but, in general, also on the wall roughness. The existing experimental results on smooth and rough walls may be understood and represented in formulas (ref. 7) directly up to the wall, and with aid of L. Prandtl's mixing-length expression. Here we are interested only in the asymptotic form for  $y > \delta_w$  which results from relation (4.2) by integration:

$$U = v^* \left[ \frac{1}{\kappa} \ln \frac{v^* y}{\nu} + C \right] \quad (4.8)$$

Therein the constant  $C$  is a function of the wall roughness.

The outer part ( $y > \delta_w$ ) has to satisfy the boundary-layer equations given in section 3.1; using the relations named in section 3.3, one may, however, neglect herein the kinematic viscosity. In flow problems of the practice, the desideratum usually is the boundary-layer flow, with the velocity at the outer edge  $U_1(x)$  and wall properties and viscosity prescribed. For theoretical investigations, the problem may be formulated differently: beside  $U_1(x)$ , the shearing stress velocity  $v^*$  is prescribed as a function of  $x$  and the desideratum is the wall condition required in order to produce this variation  $v^*(x)$ . Instead of the boundary conditions indicated in section 3.1, in this problem the following conditions for  $\lim_{y \rightarrow 0} y \rightarrow 0$  at the wall ( $y = 0$ ) must be satisfied for the outer part ( $y > \delta_w$ ) in order to guarantee the connection with expression (4.8):

$$\lim_{y \rightarrow 0} \frac{\partial U}{\partial y} = \frac{v^*}{\kappa y}; \quad V = 0; \quad E = \left(\frac{v^*}{\kappa_0}\right)^2 \quad (4.9)$$

Since on one hand the value  $\left(\frac{v^*}{U_1}\right)^2$  which corresponds to the local friction coefficient  $c_f$ ,

$$\left(\frac{v^*}{U_1}\right)^2 = \frac{\tau_0}{\rho U_1^2} = \frac{c_f}{2} \quad (4.10)$$

may be estimated quite satisfactorily, according to existing approximation formulas (for instance, ref. 10), even without exact knowledge of all boundary-layer details and varies only slowly with  $x$ , and since on the other hand the velocity profile of the outer part in case of appropriate normalization seems to be dependent on  $v^*/U_1$  only to a comparatively slight extent, as will be shown later, a treatment of the boundary layer in this manner where the outer part is simply determined with  $v^*(x)$  and  $U_1(x)$  prescribed promises some prospect of success also for the first-named problem of practice.

The presuppositions for the division into two mutually independent regions are, in most cases, satisfactorily fulfilled. This is, however, by no means self-evident and is, therefore, to be checked for the individual case. For this purpose, we add the following orders of magnitude: The thickness  $\delta_w$  is for smooth walls  $\delta_w \sim 100 \nu/v^*$ ; the pertinent Reynolds number of the turbulence for  $y = \delta_w$  is  $\sqrt{El}/\nu \sim 100$ . For pronounced wall roughness,  $\delta_w$  is determined by the dimensions of the roughnesses. According to the experiments of J. Nikuradse (ref. 8) on sand-rough pipes,  $\delta_w$  is approximately equal to the grain size of the roughness; the  $y$ -values are measured in this case from a plane of reference in which  $U$ , on the average, disappears.

## 5. EXISTENCE OF SIMILAR SOLUTIONS

### 5.1. Differential Equations and Boundary Conditions

It will now be shown that under certain assumptions so-called similar solutions exist for turbulent boundary layers, too, similar to the case of laminar boundary layers — that is, solutions for which the velocity profile along the wall is distorted only affinely. We investigate the solution of the boundary-layer equations to be expected, with neglect of the viscosity, in the range  $y > \delta_w$ . In order to satisfy the continuity

condition, we introduce for the basic flow the stream function  $\psi$  from which the components  $U$  and  $V$  are derived<sup>5</sup> by the relations

$$U = \psi_y; \quad V = -\psi_x$$

After substitution of this function into the equation of motion (3.1), there follows

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = -\frac{1}{\rho} p_x - (\overline{uv})_y \quad (5.1)$$

and the energy equation (3.4) assumes with the relations (3.17), (3.18), and (3.19) the form

$$\psi_y E_x - \psi_x E_y = -\overline{uv} \psi_{yy} - c \frac{E^{3/2}}{\kappa y} + \left( \kappa k_q y E^{3/2} \right)_{yy} \quad (5.2)$$

Finally, one obtains for  $\overline{uv}$  according to equation (3.16)

$$-uv = \kappa k \sqrt{E} y \psi_{yy} \quad (5.3)$$

It may now be shown that for velocity distributions prescribed at the outer edge of the boundary layer of the form

$$U_1 = ax^m \quad (5.4)$$

with  $a$  and  $m$  being constant quantities, and for a prescribed constant  $v^*/U_1$  there exist similar solutions for relations (5.1), (5.2), and (5.3)<sup>6</sup>. If the flow is unaffected by the viscosity, the geometric similarity of the flow pattern requires that the boundary-layer thickness for similar

<sup>5</sup>Partial derivatives with respect to  $x$  and  $y$  in this section are expressed by subscripts  $x$  and  $y$ .

<sup>6</sup>Constant  $v^*/U_1$  signifies a constant local friction coefficient. Under what circumstances and to what extent this assumption can be practically satisfied is shown in section 5.2.

solutions increase linearly with  $x^7$ . For this reason, we introduce the dimensionless variable  $\eta = y/x$  and make the statements

$$\left. \begin{aligned} \psi &= ax^{m+1}[\eta - f(\eta)] \\ E &= a^2x^{2m}\phi(\eta) \\ -\overline{uv} &= a^2x^{2m}\varphi(\eta) \end{aligned} \right\} \quad (5.5)$$

Therein  $f(\eta)$ ,  $\phi(\eta)$ ,  $\varphi(\eta)$  are only functions of the variable  $\eta$ . In order to satisfy the prescribed boundary condition (eq. (5.4)) in case of large  $y$ ,  $f(\eta)$  must for large  $\eta$  tend asymptotically toward a constant value - that is,  $\lim_{\eta \rightarrow \infty} f'(\eta) = 0$  must be true. Thus, there follows from equation (5.1) for  $\eta \rightarrow \infty$

$$-\frac{1}{\rho} p_x = a^2mx^{2m-1} \quad (5.6)$$

which results also directly from relation (5.4) and Bernoulli's equations. After substitution of equations (5.5) and (5.6) into equations (5.1) to (5.3), one obtains after division by

$$-a^2x^{2m-1}, \quad a^2x^{2m}, \quad a^3x^{3m-1}$$

respectively

$$\left. \begin{aligned} 2mf' - mf'^2 - (m+1)(\eta - f)f'' &= -\varphi' \\ (1 - f')2m\phi + (1 + m)(f - \eta)\phi' &= \\ -\varphi f'' - \frac{c}{\kappa} \frac{\phi^{3/2}}{\eta} + \kappa \frac{d^2}{d\eta^2} (\kappa_q \eta \phi^{3/2}) & \\ \varphi &= -\kappa \kappa_q \sqrt{\phi} f'' \end{aligned} \right\} \quad (5.7)$$

---

<sup>7</sup>The same results also from the momentum theorem (eq. (3.10)).

In this system of ordinary differential equations,  $x$  no longer appears explicitly so that one actually has to expect systems of solution of equations (5.5) where the boundary-layer thickness  $\delta$  increases linearly with  $x^{\frac{1}{2}}$ .

For unequivocal determination of a solution, five boundary conditions must be prescribed. In order to satisfy the three boundary conditions (relations (4.9)) at the inner edge, one has to put

$$\lim_{\eta \rightarrow 0} f'' = -\frac{\sqrt{\varphi_0}}{\kappa\eta}; \quad f(0) = 0; \quad \phi(0) = \frac{\varphi_0}{k_0^2} \quad (5.8)$$

with  $\varphi_0 = \phi(0)$ . Two conditions at the outer edge of the boundary layer are added:

$$\lim_{\eta \rightarrow \infty} ; \quad f' = 0; \quad \phi = 0 \quad (5.9)$$

The first insures that the basic flow merges with the prescribed flow; whereas, the second causes the turbulence intensity outside of the boundary layer to die out to zero.

## 5.2 Properties of the Similar Solutions

Owing to the conditions at the inner edge, there appears, in addition to the parameter  $m$  occurring in equations (5.7), as a further freely selectable quantity the value  $\varphi_0$  which like the local friction coefficient  $c_f'$  is according to relation (4.10):

$$\varphi_0 = \left(\frac{v^*}{U_1}\right)^2 = \frac{c_f'}{2} = \frac{\tau_0}{\rho U_1^2} \quad (5.10)$$

The solution of the system of equations (5.7) with the boundary conditions (relations (5.8) and (5.9)) is, therefore, a two-parameter curve family. The velocity profile of the outer part, most interesting in these solutions, may be represented in the form

$$\frac{U_1 - U}{v^*}(\eta) = \frac{f'(\eta)}{\sqrt{\varphi_0}} \quad (5.11)$$

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<sup>8</sup>For instance, the displacement thickness  $\delta_1$  according to equation (3.8) is:  $\delta_1 = x f(\infty)$ .



and is dependent on the two parameters  $m$  and  $v^*/U_1$ . Likewise, there results, of course, for the pertaining "turbulence profile," that is the plotting of the kinetic turbulence energy

$$\frac{E}{v^{*2}}(\eta) = \frac{\Phi(\eta)}{\Phi_0} \quad (5.12)$$

over  $\eta$ , also a two-parameter curve family dependent on  $m$  and  $v^*/U_1$ .

For turbulent flows in a pipe or between parallel walls, the velocity profile corresponding to equation (5.11), plotted over the wall distance  $y$  divided by the pipe diameter or, respectively, the mutual distance of the walls (so-called "center law"), is independent of the magnitude of the friction coefficient (compare, for instance, ref. 8). It seems appropriate to point out this difference between turbulent pipe and boundary-layer flow. Furthermore, attention should be called to the difference compared to the laminar boundary layers where the velocity profile is a function of only one parameter, namely  $m$ .

The solution of equations (5.7), valid only for wall distances  $y \geq \delta_w$ , must be supplemented by the wall profile (relation (4.7)) in order to obtain from it the complete velocity profiles. The condition for the continuous junction of the outer part to relation (4.7) is obtained by elimination of the quantity  $U/v^*$ , with the aid of relation (4.8), from the asymptotic form

$$\frac{U_1 - U}{v^*} = -\frac{1}{\kappa} \ln(\eta) + K\left(m, \frac{v^*}{U_1}\right) \quad (5.13)$$

resulting for small  $\eta$ -values by integration of  $f''$  according to relations (5.8). In this manner, one obtains

$$\frac{U_1}{v^*} + \frac{1}{\kappa} \ln \frac{U_1}{v^*} - K\left(m, \frac{v^*}{U_1}\right) = \frac{1}{\kappa} \ln \frac{U_1 x}{v} + C \quad (5.14)$$

The constant  $K(m, v^*/U_1)$  in equations (5.13) and (5.14) may be determined, in case of prescribed parameters  $m$  and  $v^*/U_1$ , from the system of equations (5.7).

The solutions of the outer part herein discussed have real significance only when the Reynolds number  $Re_x = \frac{U_1 x}{v}$  and the wall roughness, the effect of which is expressed in the quantity  $C$ , are such that equation (5.14) is identically satisfied for all  $x$ -values.

For extremely large Reynolds numbers, there exists a linear relation between  $C$  and the logarithm of the length characterizing the roughness (for instance, of the grain size  $k$ )<sup>9</sup>, so that the right side of equation (5.14) becomes independent of  $x$  when the grain size  $k$  is proportional to  $x$  — that is, when  $k/x = \text{Const.}$  For hydrodynamically smooth walls and for constant roughness where  $C$  is a constant, the condition (eq. (5.14)) cannot be rigorously satisfied for all  $x$ -values. This would be possible only in the case  $m = -1$  which has, however, no physical significance because then the flow separates from the wall. However, since  $x$  in equation (5.14) appears in logarithmic form, it will be permissible to regard, for sufficiently large  $x$ -values, the expression on the right side of equation (5.14) as approximately constant from  $x$ -interval to  $x$ -interval also for arbitrary  $m$ . Under this assumption, one may regard the similar solutions with practically sufficient accuracy as valid for the individual interval also for smooth walls and for walls with constant roughness. It is, however, essential that the value

$$\frac{\delta_w}{x} = \frac{\delta_w v^*}{v} \frac{U_1}{v^*} \frac{v}{U_1 x}$$

be so small that the function  $(U_1 - U)/v^*$  at the point  $y/x = \delta_w/x$  actually deviates only slightly from the asymptotic form (eq. (5.13)). Otherwise, the method selected, joining the wall law (eq. (4.7)) to the solutions obtained with neglect of the viscosity effect, does not lead to useful results.

Since the required conditions are rarely satisfied in actual cases, the similar solutions will evoke chiefly theoretical interest. One has here a type of solution of the boundary-layer equations which offers a comparatively simple survey and is thus suitable for the study of theoretical problems. It could be shown that the solutions of the outer boundary-layer part depend on  $v^*/U_1$ . On the problem regarding the extent of this influence, which is one of the next-to-most-important ones, one can, at the time, obtain information only from experiments, as will be shown in section 6.

It is perhaps necessary to point out that the only assumption for the derivation of these theoretical results was that the Reynolds number of the turbulence should be sufficiently large (except in the thin sub-layer  $\delta_w$ ) so that the viscosity may be neglected in the boundary-layer equations; aside from the customary boundary-layer simplifications no

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<sup>9</sup>For sand-rough walls, there applies, for instance, according to the experiments by J. Nikuradse (ref. 8):  $C = 8.5 - \frac{1}{\kappa} \ln \frac{v^* k}{v}$ .

restricting hypotheses were introduced. The findings thus have general validity. However, if one wants to determine the solutions of equations (5.7) numerically, one cannot forego some hypotheses; that is, one would have to introduce special formulations for  $c$ ,  $k$ , and  $k_q$ . Thus, one would, for instance, insert constant values for the factors  $c$ ,  $k$ ,  $k_q$ . This we shall not do, however. Instead, we shall attempt to obtain a survey of the solutions by empirical method by using the knowledge attained from existing test results. Since nowadays measuring series exist where the wall shearing stress was determined by a special measurement (refs. 9 and 10), a sorting of the experimental data can be undertaken with greater success than was so far possible.

## 6. EMPIRICAL BOUNDARY-LAYER PROFILES

### 6.1 Velocity Profiles

Theoretically, for the boundary layer on the plate with constant external pressure (the constant external pressure appears as special case  $m = 0$  in the system of eqs. (5.7)), a family dependent on the local friction coefficient, thus a single-parameter family, would result.

However, the plotting of  $\frac{U_1 - U}{v^*}$  against  $y/\delta$  according to F. Schultz-Grunow (ref. 11) shows that the profiles within the considered Reynolds number range may be represented with practically sufficient accuracy by a single curve. Since the boundary-layer thickness  $\delta$  used for the plotting is a quantity which can hardly be exactly defined, the expression  $yv^*/(\delta_1 U_1)$  instead of  $y/\delta$  is introduced as reference value with use of the displacement thickness  $\delta_1$ . Thereby the abscissa scale is fixed so that the integral value becomes

$$\int_0^\infty \frac{U_1 - U}{v^*} d\left(\frac{yv^*}{\delta_1 U_1}\right) = 1 \quad (6.1)$$

as one can see from a comparison with equation (3.8). In figure 1, the values  $\frac{U_1 - U}{v^*}$  for the flat plate without pressure gradient were plotted

against  $\log \frac{yv^*}{\delta_1 U_1}$ . The test points of the smooth plate according to

reference 11 fall almost into a single curve; nevertheless, close observation shows a small systematic influence of the value  $v^*/U_1$ . In contrast, the test points of the rough plate show according to reference 21 somewhat large deviations due to the greater variation of  $v^*/U_1$ . This

investigation admits the conjecture that the magnitude of the local friction coefficient is, in case of appropriate normalization of the y-scale, of only moderate influence on the velocity profile.

The boundary-layer profiles measured for variable course of pressure at the wall may be represented in the same manner. Figure 2, in which several profiles of the quoted measurements by H. Ludwig and W. Tillmann (ref. 10) are represented, shows that the pressure variation exerts a considerably stronger influence on the profile shape than  $v^*/U_1$ .

Performance of approximation calculations requires by no means knowledge of the boundary-layer profiles to the last detail; it is, on the contrary, quite sufficient to be oriented regarding the relations between the individual parameters (displacement thickness  $\delta_1$ , momentum-loss thickness  $\delta_2$ , energy-loss thickness  $\delta_3$ , and others). Various authors (refs. 10, 12, 13, 14, and 15) found empirically that, for the profiles of turbulent boundary layers, for arbitrary pressure increase, these relations are almost unequivocal - that is, that the boundary-layer profiles can be described approximately by only one parameter. Further treatment of test material will be based on this presupposition. The relation between the prescribed velocity distribution  $U_1(x)$  and the profile parameter to be defined more closely is, at first, not yet established. This relation is ascertained only by application of the momentum theorem (eq. (3.10)) and of the energy theorem (eq. (3.15)) - a method which, in principle, has been known for a long time for the calculation of laminar and turbulent boundary layers and has been very successfully applied in approximation methods. However, one should not forget that this type of single-parameter representation is no more than an approximation as opposed to the fact that, according to the theory, even in the simplest case of similar solutions a two-parameter family is to be expected. The reason for the usefulness of this approximation lies perhaps in the fact that (as was observed in the case of the flat plate without pressure gradient) the influence of one of the two parameters - namely, of the local friction coefficient - is generally probably little noticeable if the y-scale has been suitably normalized.

The next step is bringing the desired parameters into a form which permits separate consideration of the influence of the viscosity and of the wall roughness. For the momentum-loss thickness  $\delta_2$ , there applies according to equation (3.9)

$$\delta_2 = \int_0^{\infty} \frac{U}{U_1} \left(1 - \frac{U}{U_1}\right) dy = \int_0^{\infty} \left(1 - \frac{U}{U_1}\right) dy - \int_0^{\infty} \left(1 - \frac{U}{U_1}\right)^2 dy \quad (6.2)$$

which may also be written as

$$\delta_2 = \delta_1 \left( 1 - \frac{v^*}{U_1} I_1 \right) \quad (6.3)$$

The value

$$I_1 = \int_0^\infty \left[ \frac{U_1 - U}{v^*} \right]^2 d \left( \frac{y v^*}{\delta_1 U_1} \right) \quad (6.4)$$

is, under the assumptions, made practically independent of the velocity distribution for  $y < \delta_w$ . Likewise there results for the energy-loss thickness according to equation (3.13)

$$\begin{aligned} \delta_3 = \int_0^\infty \frac{U}{U_1} \left[ 1 - \left( \frac{U}{U_1} \right)^2 \right] dy &= 2 \int_0^\infty \left( 1 - \frac{U}{U_1} \right) dy - 3 \int_0^\infty \left( 1 - \frac{U}{U_1} \right)^2 dy + \dots \\ &\quad \int_0^\infty \left( 1 - \frac{U}{U_1} \right)^3 dy \end{aligned} \quad (6.5)$$

or

$$\delta_3 = \delta_1 \left[ 2 - 3 \frac{v^*}{U_1} I_1 + \left( \frac{v^*}{U_1} \right)^2 I_2 \right] \quad (6.6)$$

where

$$I_2 = \int_0^\infty \left[ \frac{U_1 - U}{v^*} \right]^3 d \left( \frac{y v^*}{\delta_1 U_1} \right) \quad (6.7)$$

also is practically independent of the profile form for  $y < \delta_w$ .

The profiles of the representation figures 1 and 2 have according to equation (5.13) for  $y \rightarrow \delta_w$  the form

$$\frac{U_1 - U}{v^*} = -\frac{1}{\kappa} \ln \frac{y v^*}{\delta_1 U_1} + K \quad (6.8)$$

with the quantity  $K$  of a different amount for every profile form. Since, on the other hand, the velocity profiles have for small  $y$ -values ( $\delta_w \leq y \ll \delta$ ) the form of expression (4.8), there results by substitution of expression (4.8) into equation (6.8)

$$\frac{U_1}{v^*} = \frac{1}{\kappa} \ln Re_1 + B \quad (6.9)$$

if

$$Re_1 = \frac{U_1 \delta_1}{\nu} \quad (6.10)$$

is the Reynolds number formed with the displacement thickness and  $B$  is

$$B = C + K \quad (6.11)$$

The quantities  $I_1$ ,  $I_2$ , and  $K$  are pure form parameters which can be immediately derived from the profile form and do not depend on the form of the wall law (eq. (4.7)) if the condition  $\delta_w \ll \delta$  is sufficiently satisfied. For a single-parameter profile family, there exists an unequivocal relation between these quantities which can be determined empirically from existing measurements. For the following considerations, we shall regard  $I_1$  as characteristic form parameter and express the others as function of  $I_1$ .

In order to obtain some sort of numerical basis for this empirical relation and thus to get away from the scatter of the test points, also to facilitate the extrapolation in the region not comprehended in the measurements, the velocity profile ( $y \geq \delta_w$ ) is represented by a simple approximation expression which starts out from the wall law (eq. (4.8))

$$U = v^* \left[ \frac{1}{\kappa} \left( \ln \frac{v^* y}{\nu} + A \frac{y}{\delta} \right) + C \right] \quad (6.12)$$

$A$  is a freely selectable constant. The thickness  $\delta$  of the boundary layer is defined for  $y = \delta$  by the condition  $U = U_1$ , so that the qualifying equation for  $\delta$  reads

$$U_1 = v^* \left[ \frac{1}{\kappa} \left( \ln \frac{v^* \delta}{\nu} + A \right) + C \right] \quad (6.13)$$

From expressions (6.12) and (6.13), there follows for the outer part of the velocity profile

$$\frac{U_1 - U}{v^*} = \frac{A}{\kappa} \left(1 - \frac{y}{\delta}\right) - \frac{1}{\kappa} \ln \frac{y}{\delta} \quad (6.14)$$

For the displacement thickness, there results hence with equation (3.8)

$$\frac{\delta_1 U_1}{\delta v^*} = \int_0^1 \frac{U_1 - U}{v^*} d\left(\frac{y}{\delta}\right) = \frac{1 + \frac{A}{2}}{\kappa} \quad (6.15)$$

Furthermore, the quadratures yield:

$$I_1 = \frac{\delta v^*}{\delta_1 U_1} \int_0^1 \left(\frac{U_1 - U}{v^*}\right)^2 d\left(\frac{y}{\delta}\right) = \frac{2 + \frac{3}{2} A + \frac{1}{3} A^2}{\kappa \left(1 + \frac{A}{2}\right)} \quad (6.16)$$

$$I_2 = \frac{\delta v^*}{\delta_1 U_1} \int_0^1 \left(\frac{U_1 - U}{v^*}\right)^3 d\left(\frac{y}{\delta}\right) = \frac{6 + \frac{21}{4} A + \frac{11}{6} A^2 + \frac{1}{4} A^3}{\kappa^2 \left(1 + \frac{A}{2}\right)} \quad (6.17)$$

Thereby the relation between  $I_1$  and  $I_2$  is given by a parameter representation which is plotted in figure 3 for  $\kappa = 0.4$  and compared<sup>10</sup> with the quoted measurements by H. Ludwig and W. Tillmann (ref. 10) and F. Schultz-Grunow (ref. 11).

For the quantity  $B$  in equation (6.9),

$$B = C + \frac{A}{\kappa} - \frac{1}{\kappa} \ln \frac{1 + \frac{A}{2}}{\kappa} \quad (6.18)$$

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<sup>10</sup>The relatively large scatter of the test points for small  $I_1$ -values is without practical significance because the term in equation (6.6) dependent on  $I_2$  contributes only a very small percentage to  $\delta_3$ ; the scatter is explained by the fact that the experimental  $I_1$ - and  $I_2$ -values were not obtained directly by quadratures but were calculated backward from the  $\delta_1$ -,  $\delta_2$ -, and  $\delta_3$ -values determined by quadratures with use of the experimentally ascertained  $v^*/U_1$  from equations (6.3) and (6.6).

would result from expressions (6.12) and (6.13). Although the expression (6.12) renders the velocity profile on the average quite satisfactorily, deviations do appear in details, which take effect chiefly in the quantity  $K$  according to equation (6.8). Therefore, the  $B$ -value is not satisfactorily represented by equation (6.18); the modified form

$$B = C + 0.82 \frac{A}{\kappa} - \frac{1}{\kappa} \ln \frac{1 + \frac{A}{2}}{\kappa} \quad (6.19)$$

is more appropriate, as shown by the comparison with measurements represented in figure 4 for  $C = 5.2$  and  $\kappa = 0.4$ .

Figures 3 and 4 may be regarded as a confirmation for the usefulness of the representation dependent only on the form parameter  $I_1$  and of the expression (6.12).

## 6.2. Turbulence Profiles

If one considers use of the energy theorem (3.15), one needs data one cannot obtain from the velocity profile alone. In order to ascertain the magnitude of the energy flow, one requires the turbulence profile which in a dimensionless plotting corresponding to figures 1 and 2 is represented as  $E/v_*^2$  over  $(yv_*)/(\delta_1 U_1)$ . Herein  $E/v_*^2$  tends in the neighborhood of the wall  $y \rightarrow \delta_w$  toward the universally valid value given by expression (4.3) and decreases for  $y < \delta_w$  very rapidly to zero. Although it is fundamentally possible to determine, with known hot-wire arrangements, the quadratic mean values of all three fluctuation components experimentally and hence, according to relation (2.1),  $E$  numerically, evaluable measurements exist only for the component  $u$  which, it is true, yields the most essential contribution to  $E$ . The

longitudinal oscillation profiles  $\sqrt{u^2}/v_*$  represented in figures 5 and 6 were measured by means of the turbulence-measuring device of W. Tillmann (refs. 16 and 21) developed by H. Schuh. The conjecture following from the universal boundary-layer flow and  $\delta_w \ll \delta$  that the  $\sqrt{u^2}/v_*$  tends toward a universal value in the same manner as  $E/v_*^2$  for  $y \rightarrow \delta_w$ , is only insufficiently confirmed by these measurements. The reason probably lies in inadequacies of measuring technique.

According to the considerations of section 5, two-parameter curve families for  $E/v_*^2$  would result for the similar solutions. If, however, the influence of the one parameter  $v_*/U_1$  on the velocity profile is small, which is probable according to the preceding section, one may



conclude with some certainty from consideration of the third part of equation (5.7) that this parameter exerts only slight influence on the turbulence profile as well. Figure 6 confirms the correctness of this reasoning for the longitudinal-oscillation profiles of the plate flow without pressure gradient for smooth and rough surfaces. Thus, the conjecture suggests itself that the quantities of interest in turbulence profiles may, like the parameters of the velocity profiles, approximately be described by an unequivocal relation to the form parameter  $I_1$ . This assumption is taken as the basis of the further investigations.

As could be determined so far, the variation of the turbulence energy flow mostly does not make a very significant contribution for two-dimensional boundary-layer flow, so that a somewhat liberal treatment of this influence seems, as a rule, permissible. From a few older measurements by H. Reichardt (ref. 17) in a rectangular channel, by H. C. H. Townend (ref. 18) in a square pipe, and by A. Fage (ref. 19) in a circular pipe, the order of magnitude of the v- and w-variation components can be estimated. Near the wall, the v-component in particular is essentially smaller than the u-component; at larger distance from the wall, the magnitudes of the v- and w-components approach that of the u-component.

According to definition of the integral expressions practically independent of the wall law (eq. (4.7))

$$I_{T_1} = \int_0^\infty \frac{E}{v_*^2} d\left(\frac{yv_*}{\delta_1 U_1}\right) \quad \text{and} \quad I_{T_2} = \int_0^\infty \frac{U_1 - U}{v_*} \frac{E}{v_*^2} d\left(\frac{yv_*}{\delta_1 U_1}\right) \quad (6.20)$$

the turbulence-energy flow is determined to be

$$\int_0^\infty UE \, dy = \delta_1 U_1^2 v_* \left( I_{T_1} - \frac{v_*}{U_1} I_{T_2} \right) \quad (6.21)$$

For the presupposed single-parameter condition,  $I_{T_1}$  and  $I_{T_2}$  are only functions of  $I_1$ . According to the existing data, the relation

$$\int_0^\infty UE \, dy = 0.65 \delta_1 U_1^2 v_* \quad (6.22)$$

seems to be useful for the estimation independently of  $I_1$ ; it is, therefore, taken as the basis for further evaluations and calculations.

## 6.3. Dissipation Function

For determination of the dissipation function  $D$  occurring in the energy theorem (eq. (3.15)) according to expression (3.14), one will again attempt by means of the results represented in section 4 to express separately the influence of the viscosity and of the wall roughness. For this purpose, one may determine  $S$  from equation (3.4) and perform the quadrature for the part near the wall ( $0 \leq y \ll \delta$ ) if one puts  $\tau/\rho = v^* = \text{Const.}$  and the left side of equation (3.4) equal to zero which is admissible for small wall distances. One then obtains for  $y \ll \delta$

$$\int_0^y S dy' = v^{*2} U(y) - Q(y) \quad (6.23)$$

With  $U$  according to relation (4.8) and  $Q$  according to relation (4.4), there results, hence, if the upper integration limit lies in the region  $\delta_w \leq y \ll \delta$

$$\int_0^y S dy' = v^{*3} \left( \frac{1}{\kappa} \ln \frac{v^* y}{v} + C + \kappa \frac{k_{q0}}{k_0^3} \right) \quad (6.23a)$$

For the outer part of the boundary layer  $y \geq \delta_w$ , one obtains with the relation (3.18)

$$\int_y^\infty S dy' = \frac{v^{*3}}{\kappa} \int_{yv^*/\delta_1 U_1}^\infty c \frac{(E/v^{*2})^{3/2}}{(y'v^*)/(\delta_1 U_1)} d\left(\frac{y'v^*}{\delta_1 U_1}\right) \quad (6.24)$$

Since, for  $\delta_w \leq y \ll \delta$ ,  $E/v^{*2} = 1/k_0^2$  is valid according to expression (4.3) and  $c_0 = k_0^3$  according to relation (4.6), there results from equation (6.24)

$$\int_y^\infty S dy' = v^{*3} \left( J_S - \frac{1}{\kappa} \ln \frac{yv^*}{\delta_1 U_1} \right) \quad (6.24a)$$

wherein the value of the integral expression

$$J_S = \frac{1}{\kappa} \int_{yv^*/\delta_1 U_1}^\infty c \frac{(E/v^{*2})^{3/2}}{(y'v^*)/(\delta_1 U_1)} d\left(\frac{y'v^*}{\delta_1 U_1}\right) + \frac{1}{\kappa} \ln \frac{yv^*}{\delta_1 U_1} \quad (6.25)$$

because of  $c_0(E/v^2)^{3/2} = 1$  is independent of the lower integration limit  $y$  if it lies in the range  $\delta_w \leq y \ll \delta$ . From equations (6.23a) and (6.24a), one finally obtains for  $D$  the form

$$D = \int_0^\infty S \, dy = v^3 \left( \frac{1}{\kappa} \ln Re_1 + G \right) \quad (6.26)$$

with

$$G = C + J_s + \kappa \frac{k_{q0}}{k_0^3} \quad (6.27)$$

We now again assume that for our single-parameter velocity and turbulence profiles  $J_s$ , and, therewith for equal wall properties,  $G$  as well, is only a function of the parameter  $I_1$  described by expression (6.4). Since nothing is known regarding the behavior of the function  $c$ , except for the region  $\delta_w \leq y \ll \delta$ ,  $J_s$  cannot be calculated from expression (6.25), even if the turbulence profile is known. Thus, there remains only the possibility of calculating the function  $D$  by differentiation, by means of insertion of the experimentally determinable quantities into the energy equation (3.15); this method suffers, however, from serious uncertainties. The measuring series of F. Schultz-Grunow (ref. 11) on the plate without pressure gradient could be evaluated quite satisfactorily according to this method. The result represented in figure 7 is to be evaluated as a satisfactory confirmation of the correctness of the relation (6.26).

If boundary-layer measurements with pressure increase are made in a wind tunnel of rectangular cross section, the occurring secondary flows represent a disturbance, as shown by a very careful investigation by W. Tillmann (ref. 16). These secondary flows have the effect that the flow is not two-dimensional (as had been assumed in the derivation of eq. (3.15)) but at the location of the measurement usually convergent with respect to the planes parallel to the wall. A. Kehl (ref. 13) has shown how to consider in the momentum theorem (eq. (3.10)) a convergence or divergence influence. In a similar manner, the energy theorem (eq. (3.15)) for wedge-shaped flow may be ascertained; it is given herein without derivation (compare fig. 8)

$$\frac{1}{2} \frac{d}{dx} (U_1^3 \delta_3) + \frac{1}{2} \frac{U_1^3 \delta_3}{x_0 + x} = D + \frac{d}{dx} \int_0^\infty U E \, dy + \frac{\int_0^\infty U E \, dy}{x_0 + x} \quad (6.28)$$

In the evaluation of the quoted measuring series by H. Ludwig and W. Tillmann (ref. 10), the following method was applied: The mean measure of convergence  $-\frac{1}{x_0 + x}$ , which is to express summarily the

secondary-flow effects in equation (6.28), was estimated with the aid of the momentum theorem given by A. Kehl since all quantities appearing in it, with exception of the measure of convergence, were determined experimentally. This measure of convergence then was introduced into the energy theorem (eq. (6.28)) and, thus, the function  $D$  was determined. In this manner, it was possible to eliminate at least approximately the effect of the secondary flows. Aside from these measurements, four further measuring series performed by W. Tillmann in the same wind tunnel but not published were treated in the same manner<sup>11</sup>.

The result of this evaluation, which for the first time conveys an indication for the magnitude of the dissipation function as a function of the profile shape, is shown in figure 9. The scatter is sometimes quite considerable; however, on the whole, the test points are grouped fairly satisfactorily about a mean curve. Greater accuracy was hardly to be expected in view of the circumstances described<sup>12</sup>. For large  $I_1$ -values, the results may be approximatively rendered by

$$G = 7.5(I_1 - 8.2) \quad (6.29)$$

In order to make a more reliable determination of the dissipation function (which is very important for the development of approximation methods for the calculation of turbulent boundary layers), measurements would be required for which by avoidance of secondary flows easily surveyable flow conditions exist. Measurements in a rotationally symmetrical wind tunnel probably ensure clear conditions. These measurements would have to include a very exact experimental determination of the turbulence profiles, for instance, by hot-wire measurements. The reason why turbulence measurements of boundary layers have been performed comparatively rarely can probably be found, amongst other reasons, in that so far no immediate need for quantitative measurements of this kind existed.

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<sup>11</sup>The magnitude of the wall shearing stress which had not been experimentally determined in these measurements could be estimated by means of the relations given by equations (6.16) and (6.19) and figure 4, respectively.

<sup>12</sup>Particularly uncertain are the end points of the individual measuring series which are denoted by "E" in figures 7 and 9 because the variation of the curve to the differentiated is not exactly fixed at the end of each measuring series.

The energy equation in the form (3.15) will probably stimulate carrying out of further turbulence measurements.

It would mean an essential progress in the determination of the dissipation function if not only the wall shearing stress but also the entire "shearing stress profile" could be determined experimentally. Attempts to measure the mean value of the product  $\overline{uv}$ , which is according to expression (3.3) decisive for the shearing stress by means of hot-wire probes, were made by H. Reichardt (ref. 17) and H. K. Skramstad<sup>13</sup>. Besides, H. Reichardt (ref. 20) has tried to measure mechanically the mean value  $\overline{uv}$  with an angle probe. Further development of methods of this type will be of great advantage for the investigation of turbulent boundary layers.

## 7. NUMERICAL ESTIMATION OF THE SIMILAR SOLUTIONS

The relations determined from the existing test material may serve for developing approximation methods for calculation of turbulent boundary layers with arbitrary pressure gradient. Here we shall use them for quantitatively estimating the conditions for the similar solutions treated in section 5 with the aid of the momentum equation (3.10) and the energy equation (3.15).

In consequence of the results of section 5, according to which the velocity distribution is prescribed in the form of a power law (relation (5.4)) and the boundary-layer thickness  $\delta$  increases linearly with  $x$ , we make the statements

$$\left. \begin{aligned} U_1 &= ax^m \\ \delta_2 &= bx \end{aligned} \right\} \quad (7.1)$$

therein  $a$  and  $b$  are quantities independent of  $x$ . If one takes into consideration that the form parameters of the velocity profiles  $H_{12} = \delta_1/\delta_2$  and  $H_{32} = \delta_3/\delta_2$  are, according to presupposition, also independent of  $x$ , there results by substitution of equations (7.1) into the momentum equation (3.10) and after division by the value  $(ax^m)^2$

$$(2m + 1)b + mbH_{12} = \left( \frac{v^*}{U_1} \right)^2 \quad (7.2)$$

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<sup>13</sup>National Bureau of Standards, Washington, D. C., USA.

From the energy theorem (eq. (3.15)), one obtains in the same manner, with use of relation (6.22)

$$b \left( \frac{H_{32}}{2} - 0.65 \frac{v^*}{U_1} H_{12} \right) (3m + 1) = \left( \frac{v^*}{U_1} \right)^3 \frac{D}{v^{*3}} \quad (7.3)$$

For the momentum loss thickness  $\delta_2$ , there follows from equation (7.2)

$$\frac{\delta_2}{x} = b = \frac{(v^*/U_1)^2}{(2m + 1 + mH_{12})} \quad (7.4)$$

Since the calculation of the boundary layer for a prescribed velocity distribution is troublesome, we choose a more convenient method and determine for prescribed values of the boundary-layer profile the pertinent velocity variation along  $x$ , that is, the exponent  $m$ . For this purpose, equation (7.3) is, after elimination of  $b$  and with the aid of relation (7.4), solved with respect to  $m$ :

$$m = - \frac{H_{32}/2 - (v^*/U_1) (D/v^{*3} + 0.65H_{12})}{3H_{32}/2 - (v^*/U_1) [(2 + H_{12})D/v^{*3} + 1.95H_{12}]} \quad (7.5)$$

This equation is evaluated by calculation of the quantity  $v^*/U_1$  for assumed values of the Reynolds number  $Re_1 = U_1\delta_1/\nu$  and of the profile parameter  $I_1$  with the aid of the relation (6.9) and figure 4. From equation (6.3) then results  $H_{12} = \delta_1/\delta_2$ . With equation (6.6) and figure 3, one may then proceed to calculate  $\delta_3/\delta_1$  and therewith  $H_{32} = \delta_3/\delta_2$ . Equation (6.26) and figure 9 make, furthermore, the determination of  $D/v^{*3}$  possible. With these quantities, it is finally possible to determine from equation (7.5) the exponent  $m$ , from relation (7.4) the momentum-loss thickness  $\delta_2/x$  referred to  $x$ , and from relation (4.10) the friction coefficient  $c_f' = 2(v^*/U_1)^2$ . In figure 10, the results of such a calculation are compiled for three different Reynolds numbers, with the conditions of smooth walls taken as a basis although they do not exactly satisfy the presuppositions of the similar solutions.

It is an interesting result that a physically meaningful solution does not exist for all  $m$ -values. This state of affairs is not immediately evident from the system of equations (5.7); it follows, however,

at once from the momentum theorem. If  $\delta_2$  and  $c_f'$  are to be positive, one will according to equations (3.8) and (3.9), because of  $U/U_1 \leq 1$ , always have  $\delta_1 > \delta_2$ ; thus,  $H_{12} > 1$ . Negative values of  $\delta_2$  and  $c_f'$  can occur only when reverse flow appears near the wall. However, in this case, the boundary-layer theory loses its physical significance since the flow separates from the wall. According to relation (7.4),  $m$  must therefore be greater than  $-1/3$ . Figure 10 shows that the separation is to be expected approximately in the range of  $m = -0.2$ .<sup>14</sup> For comparison, it should be mentioned that, for the corresponding similar solutions of the laminar boundary layers, the separation takes place at  $m = -0.091$ . This confirms the well-known empirical fact that turbulent boundary layers can overcome a larger pressure increase than laminar ones.

Another noteworthy result is the dependence of the profile parameter  $H_{12}$  coordinated to a certain  $m$ -value on the Reynolds number. The smaller the Reynolds number, the larger is  $H_{12}$ . This dependence comes about chiefly due to the fact that the wall law (eq. (4.7)) corresponding to the respective Reynolds number is adapted, according to equation (6.9), to the single-parameter profile of the outer part ( $y \geq \delta_w$ ) which is independent of Reynolds number. In this manner, the first-order effect of the Reynolds number on the velocity profile is included so that figure 10 actually is based on a two-parameter profile family. The dependence of the outer profile parts on  $v^*/U_1$ , theoretically proved in section 5, may be regarded as a Reynolds number effect of the second order; this effect was not accurately expressed in the calculation for figure 10. In an investigation by A. E. von Deonhoff and N. Tetervin (ref. 15) who calculated similar solutions with the aid of the approximation method for calculation of turbulent boundary layers indicated by them, a universal relation was found to exist between the exponent of the velocity law (eq. (7.1)) and the parameter  $H_{12}$ ; thus no dependence on  $Re$  existed. However, as figure 10 shows, the influence of the Reynolds number, the expression of which became possible only after one had succeeded in the experimental determination of the wall shearing stress, is rather important for the relation between  $H_{12}$  and  $m$ .

It need not be explained further that corresponding calculations may be carried out for flows at rough walls as well. For this purpose, one has merely to perform a conversion of the values  $B$  and  $G$  introduced in section 6 corresponding to the modified constant  $C$ . In principle, however, these calculations would not offer anything new.

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<sup>14</sup>The test data at disposal is insufficient for exact determination of the  $m$ -value corresponding to separation.

## 8. SUMMARY

As equations of the turbulent boundary layer, this report indicates the customary equation of motion, the continuity equation, and, in addition, a balance for the kinetic turbulence energy from which one may derive for approximation calculations besides the known momentum theorem also an energy theorem for turbulent boundary layers.

Under the assumption (frequently confirmed by test observations) that the influence of the kinematic viscosity and of the wall roughness takes immediate effect only in a very thin layer  $\delta_w$  at the wall, there exists within the turbulent boundary layer a region ( $\delta_w \leq y \ll \delta$ ) in which a universal flow prevails which is determined by the magnitude of the wall shearing stress but, for the rest, is not influenced either by the wall conditions or the velocity distribution  $U_1(x)$  prescribed at the outer edge. The presence of this universal boundary-layer flow enables the division of the boundary layer into a part near the wall ( $0 \leq y \leq \delta_w$ ) which is affected only by the viscosity and the wall properties and into an outer part ( $y \geq \delta_w$ ) independent of the viscosity in which the flow is essentially determined by the velocity distribution prescribed at the outer edge. The flows in these two parts show a mutual influence only insofar as the asymptotic behavior of the inner flow represents a boundary condition for the outer flow.

With the aid of the indicated boundary-layer equations, it can be proved that, for a prescribed velocity distribution  $U_1 = a x^m$  and a local friction coefficient which is almost independent of  $x$ , similar solutions exist also for turbulent boundary layers; these solutions depend on two parameters - the exponent  $m$  and the local friction coefficient. The boundary-layer thickness increases linearly with  $x$ .

With consideration of the findings obtained, the evaluation of existing test data then yields empirically the relations between the various quantities required for application of the momentum theorem and the energy theorem. Finally, the established relations are used to perform, with the friction laws valid for smooth walls taken as a basis, approximation calculations for the similar solutions.

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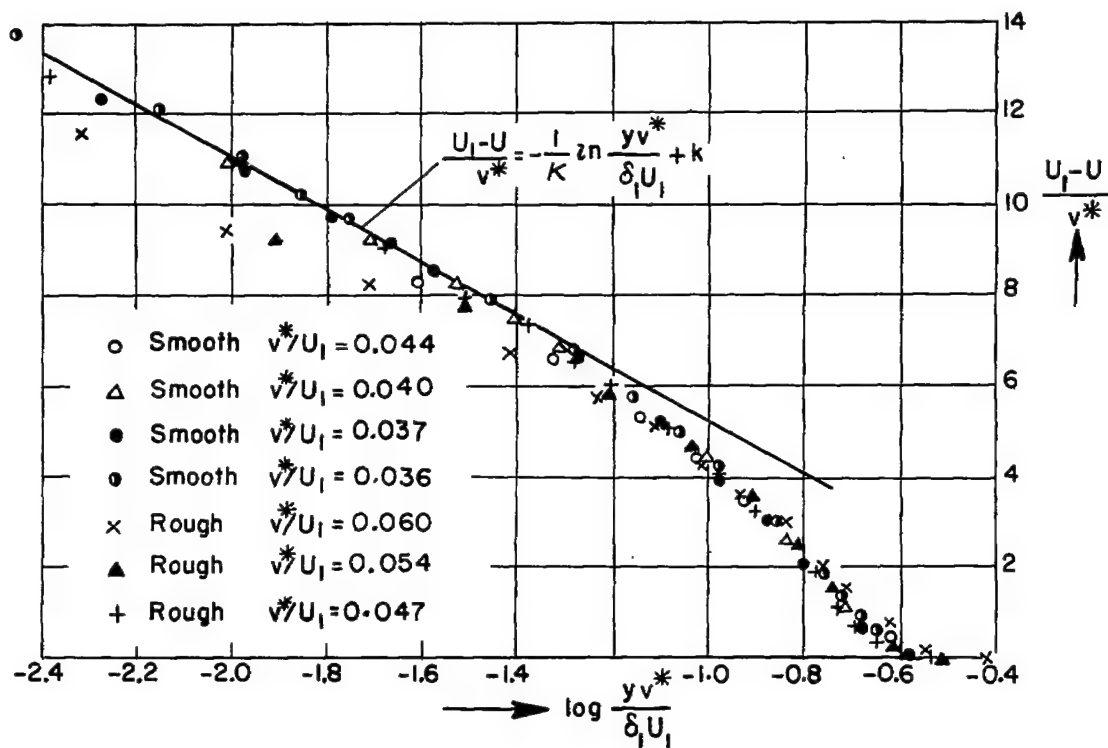


Figure 1.- Velocity profiles for constant pressure according to measurements of F. Schultz-Grunow (reference 11) on smooth walls and of W. Tillmann (reference 21) on rough walls.

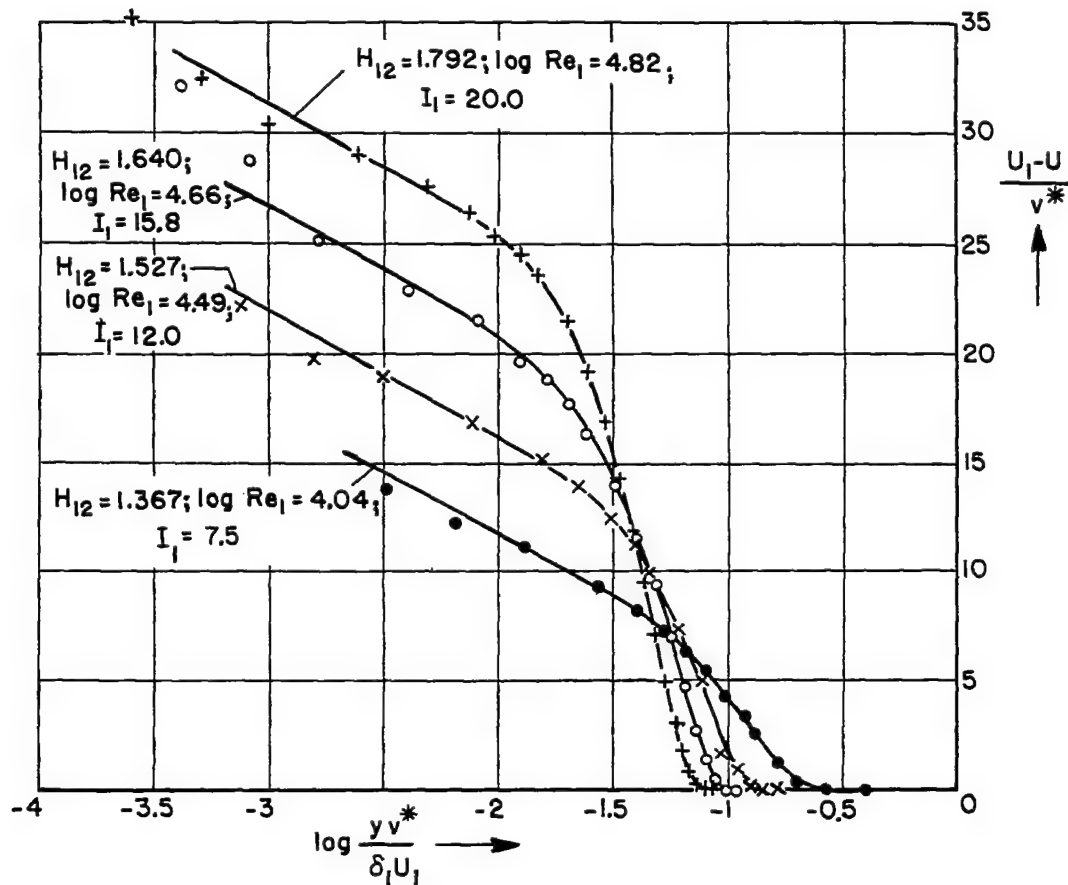


Figure 2.- Velocity profiles in case of pressure increase according to measurements of H. Ludwig and W. Tillmann (reference 10).

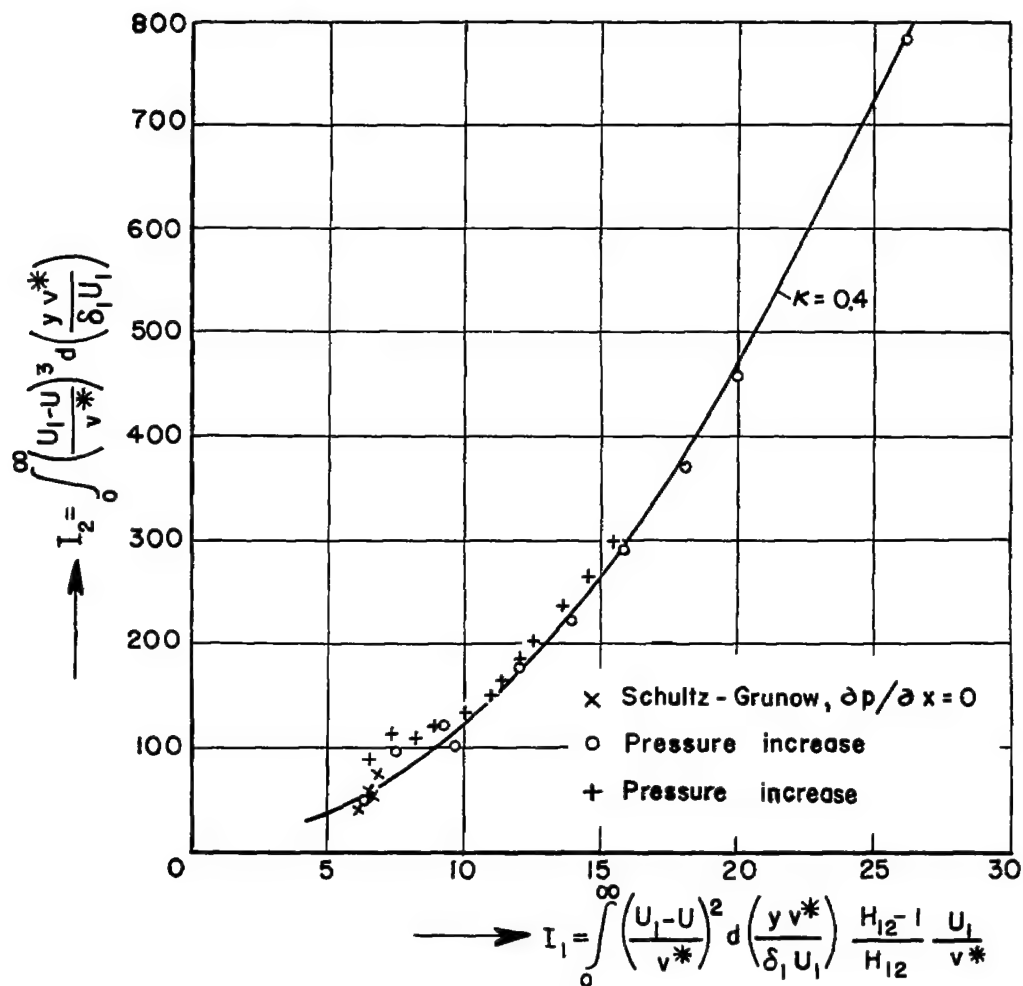


Figure 3.- Relation between the profile parameters  $I_1$  and  $I_2$  according to relations (6.16) and (6.17) and according to tests (references 10 and 11).  $H_{12} = \delta_1 / \delta_2$ .

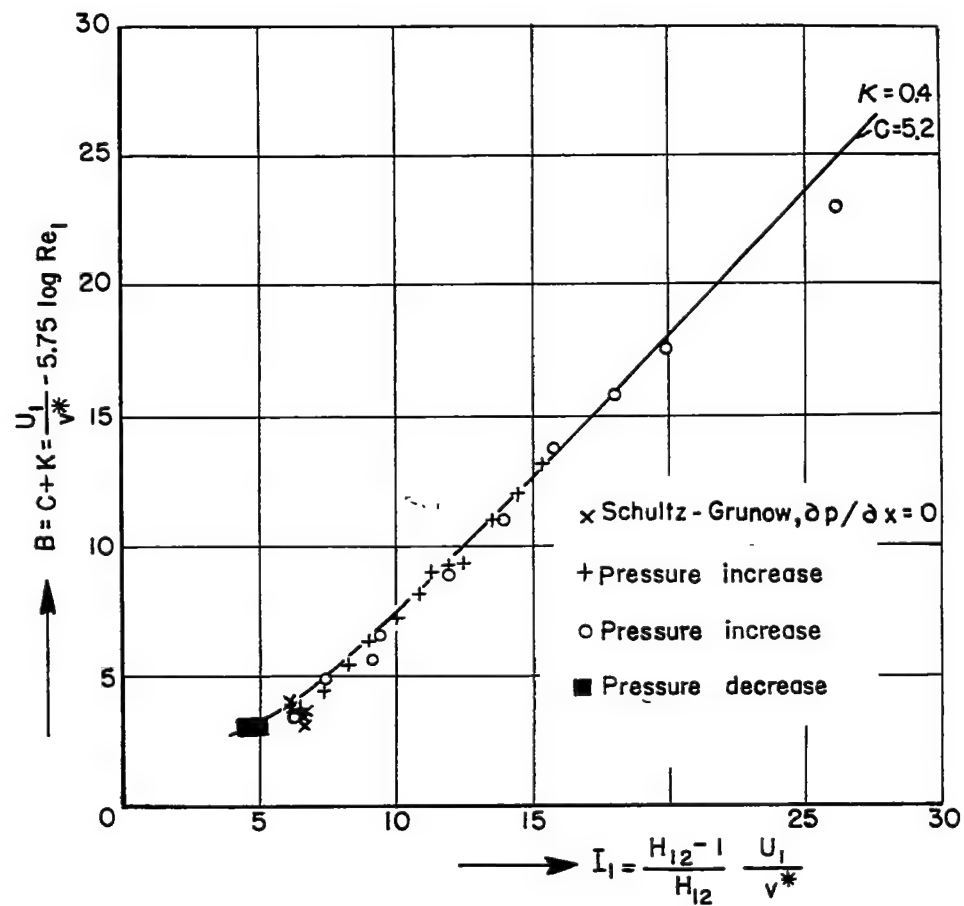


Figure 4.- Connection between the relation  $v^*/U_1$ , the Reynolds number  $Re_1$ , and the profile parameter  $I_1$  according to relations (6.16) and (6.19) and according to tests (references 10 and 11) on smooth walls.

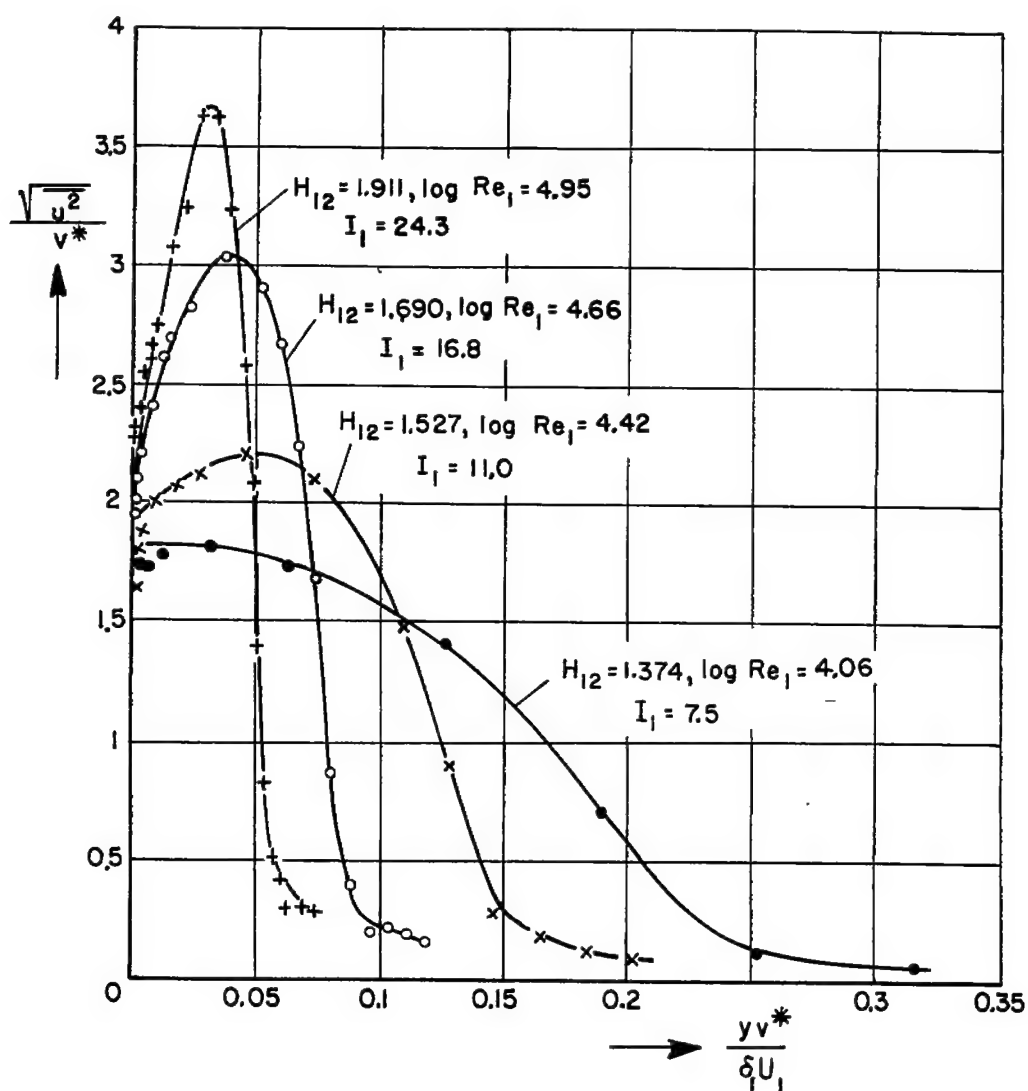


Figure 5.- Longitudinal-variation profiles for pressure increase according to measurements by W. Tillmann (reference 16).

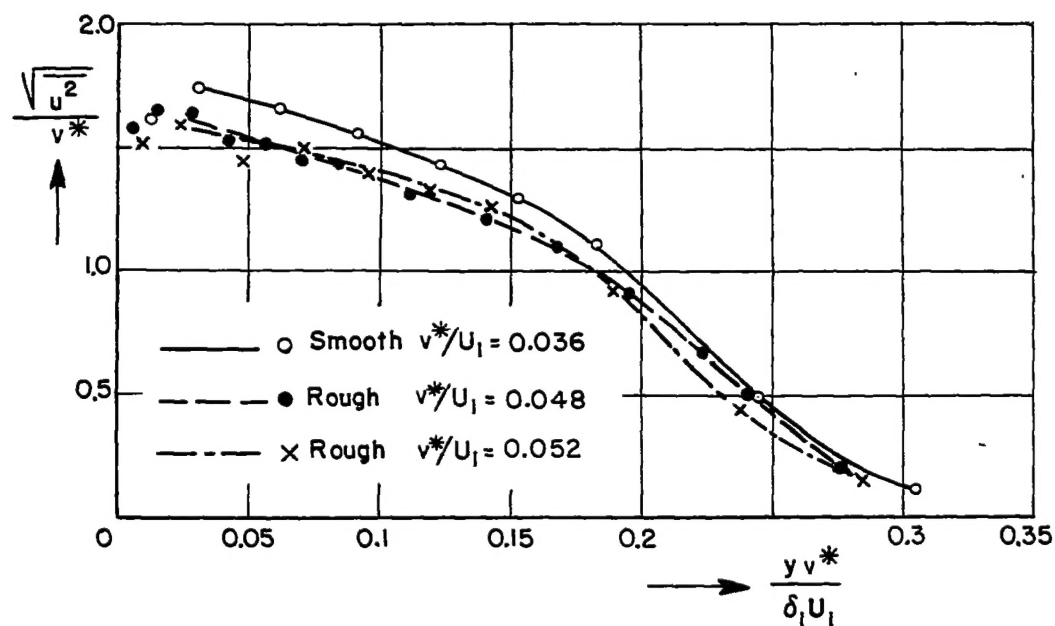


Figure 6.- Longitudinal-variation profiles for constant pressure according to measurements by W. Tillmann (reference 21) on smooth and rough walls.



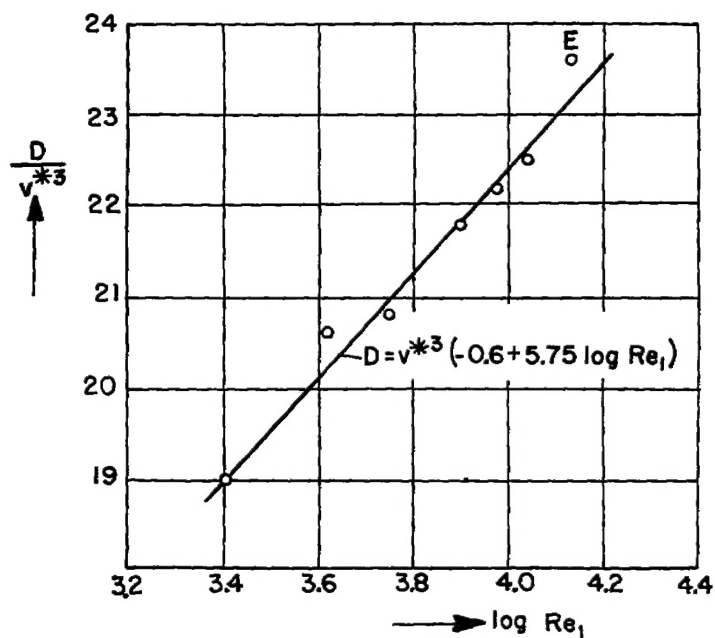


Figure 7.- Dissipation function for the boundary layer without pressure gradient as a function of the Reynolds number according to measured results by F. Schultz-Grunow (reference 11). E, end point of the measuring series.

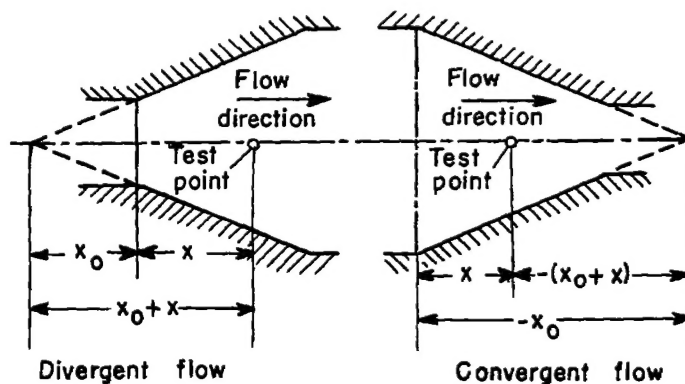


Figure 8.- Designations regarding the energy theorem (equation (6.28)).

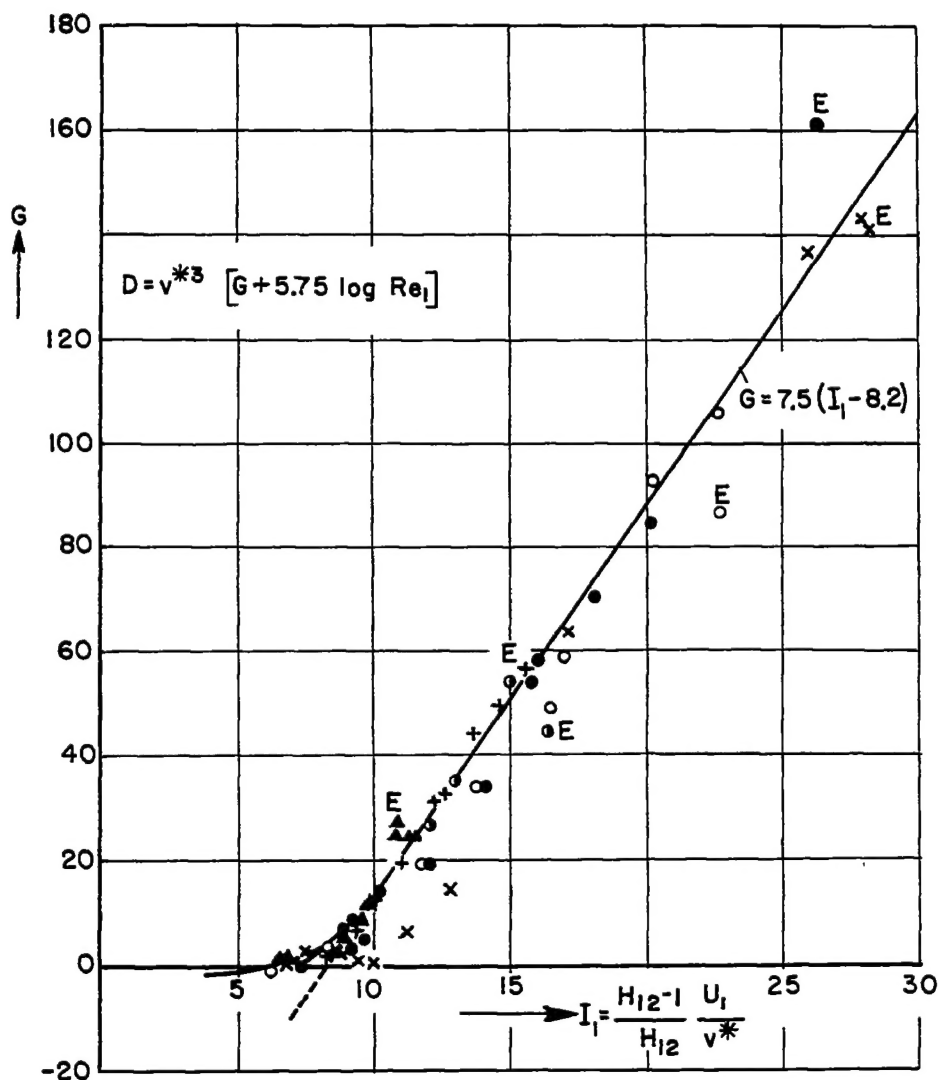


Figure 9.- Relation between the dissipation function according to relation (6.26), the Reynolds number  $Re_1$ , and the profile parameter  $I_1$  according to evaluation of measured results on smooth walls. Each series of measurements is characterized by a special symbol. E, end points of the individual measuring series.

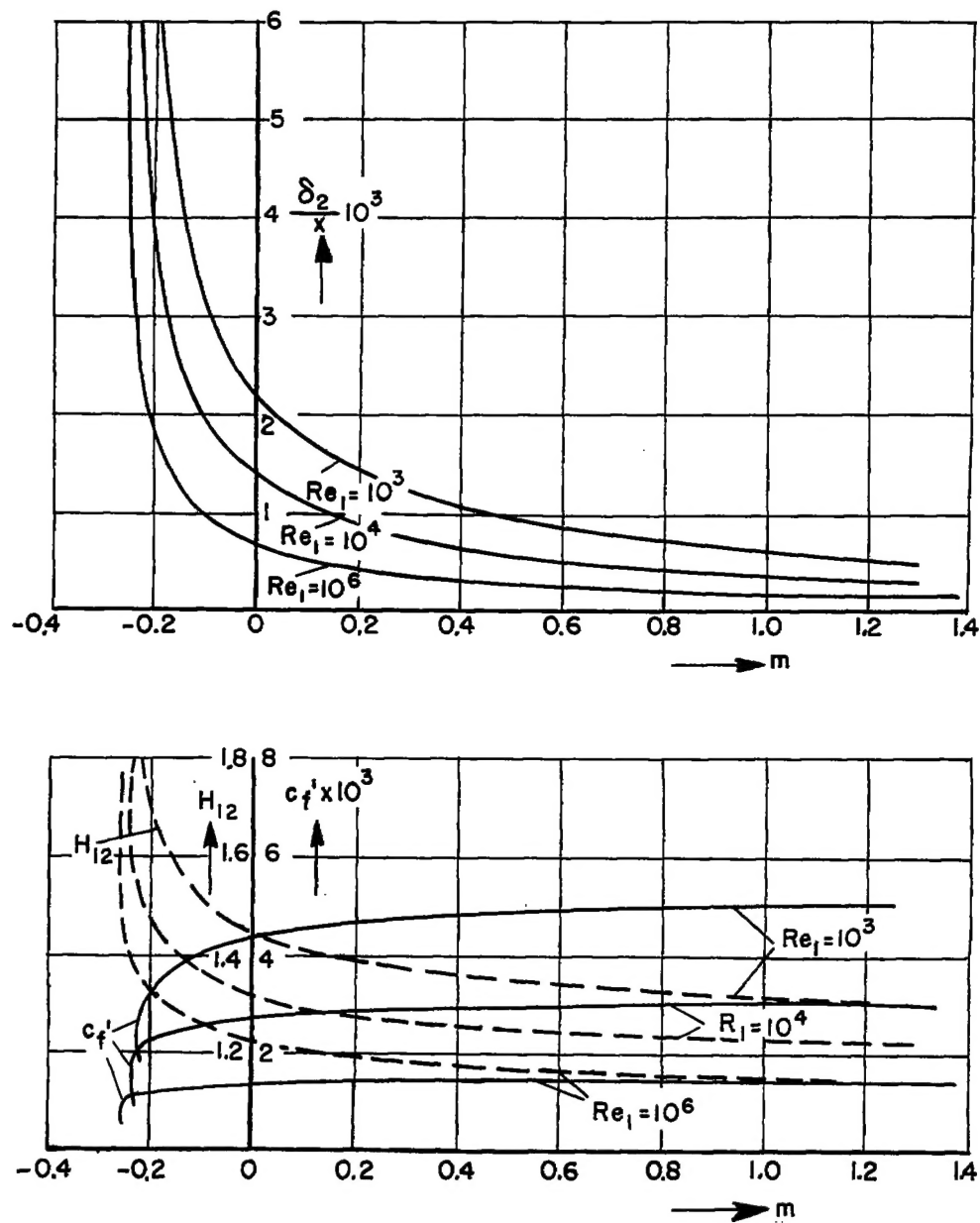


Figure 10.- Similar solutions of the equations for the turbulent boundary layer on smooth walls.